

Admixture and Drift in Oscillating Fluid Flows

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The motions of a passive scalar \hat{a} in a general high-frequency oscillating flow are studied. Our aim is threefold: (i) to obtain different classes of general solutions; (ii) to identify, classify, and develop related asymptotic procedures; and (iii) to study the notion of drift motion and the limits of its applicability. The used mathematical approach combines a version of the two-timing method, the Eulerian averaging procedure, and several novel elements. Our main results are: (i) the scaling procedure produces two independent dimensionless scaling parameters: inverse frequency $1/\omega$ and displacement amplitude δ ; (ii) we propose the *inspection procedure* that allows to find the natural functional forms of asymptotic solutions for $1/\omega \rightarrow 0, \delta \rightarrow 0$ and leads to the key notions of *critical*, *sub-critical*, and *supercritical asymptotic families* of solutions; (iii) we solve the asymptotic problems for an arbitrary given oscillating flow and any initial data for \hat{a} ; (iv) these solutions show that there are at least three different drift velocities which correspond to different asymptotic paths on the plane $(1/\omega, \delta)$; each velocity has dimensionless magnitude $O(1)$; (v) the obtained solutions also show that the averaged motion of a scalar represents a pure drift for the zeroth and first approximations and a drift combined with *pseudo-diffusion* for the second approximation; (vi) we have shown how the changing of a time-scale produces new classes of solutions; (vii) we develop the two-timing theories of a drift based on both the *GLM*-theory and the dynamical systems approach; (viii) examples illustrating different options of drifts and pseudo-diffusion are presented.

1. Introduction

The transport of a scalar field \hat{a} by an oscillating velocity field $\hat{\mathbf{u}}$ represents a classical problem of fluid dynamics. One can find a number of models, physical effects, and applications in Taylor (1953), Moffatt (1983), Hydon & Pedley (1993), Frisch (1995), Moffatt (1998), Magar & Pedley (2005), Leal (2007). The transport of a scalar represents a paradigm problem for the use of the dynamical systems approach in fluid dynamics (see Aref (1984), Ottino (1989), Solomon, Lee, and Fogelman (2001), Samelson & Wiggins (2006)) and for the studies of turbulence (see Sreenivasan (1991), Warhaft (2000), Balk (2002)). It is well-known that even in purely oscillating flows the motion of a material particle consists of two parts: oscillating and non-oscillating. The latter is known as a *drift*, see Stokes (1847), Maxwell (1870), Lamb (1932), Longuet-Higgins (1953), Darwin (1953), Hunt (1964), Batchelor (1967), Schlichting (1979), Lighthill (1978a), Andrews & McIntyre (1978), Craik (1982), Grimshaw (1984), Craik (1985), Benjamin (1986), Eames & McIntyre (1999), Eames, Belcher, and Hunt (1994). Different notions of a drift are used in physics, see *e.g.* Vergassola & Avellaneda (1997), Chierchia & Gallavotti (1994).

In our paper we consider a high-frequency asymptotic problem of a scalar transport by a given oscillatory velocity field. A characteristic length L for both $\hat{\mathbf{u}}$ and

\hat{a} is arbitrary. We use a simple and straightforward formulation of the problem that allows to study the general properties of solutions without the consideration of particular applications. The evolution of \hat{a} is exploited for the study of the concept of a drift motion and the limits of its applicability; this approach allows a great degree of flexibility and has not been used before. We employ the mathematical method which comprises the two-timing approach (Nayfeh (1973)), the fast-time averaging operation, and several novel elements; we call it *two-timing-and-averaging method (TTAM)*. The interplay of the two-timing form of solutions and a fast-time-averaging operation was exploited by many authors. The well-known applications in classical mechanics belong to Kapitza (1951a), Kapitza (1951b), Landau & Lifshits (2000). A simplified version of the two-timing method has been extensively used in mechanical engineering and is known as *vibrational mechanics*, see *e.g.* Blekhman (2000), Blekhman (2004). However Kapitza (1951a), Kapitza (1951b), Landau & Lifshits (2000), Blekhman (2000), Blekhman (2004) did not use (and even deliberately denied) the underlying asymptotic nature of the method. This nature has been emphasised and exploited by Vladimirov (2005), Yudovich (2006), Vladimirov (2008) in a particular form of *TTAM* used in this paper.

Let us outline here the content and the main results:

In *Sect.2* we formulate the problem, introduce two-timing and dimensionless versions of the governing PDE. The two independent dimensionless scaling parameters are inverse frequency $1/\omega$ and displacement amplitude δ of oscillations. Our asymptotic procedure implies both $1/\omega \rightarrow 0$ and $\delta \rightarrow 0$. It produces the multiplicity of available asymptotic paths on the plane $(1/\omega, \delta)$; we parameterize these paths with a single small parameter $\varepsilon_\alpha = 1/\omega^\alpha$, $\alpha = \text{const} > 0$. This multiplicity gives us new opportunities for the study of a drift. At the end of the section we consider an important case when $\hat{\mathbf{u}}$ does not contain any slow time-scale T : in this case we show the presence of an infinite number of solutions with a range of different T .

In *Sect.3* we develop *the inspection procedure* for the governing equations. It leads to classification of different asymptotic solutions and to the key notions of *critical*, *sub-critical*, and *super-critical* asymptotic families of solutions. For a critical asymptotic family: (i) the small parameter is $\varepsilon_{1/2}$ (or $\alpha = 1/2$); and (ii) the correlation $\langle \tilde{\mathbf{u}}\tilde{\mathbf{a}} \rangle$ between the oscillatory parts $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{a}}$ of $\hat{\mathbf{u}}$ and $\hat{\mathbf{a}}$ enters the $O(1)$ approximation of the averaged governing equation. For a super-critical family $\alpha < 1/2$ and again $\langle \tilde{\mathbf{u}}\tilde{\mathbf{a}} \rangle = O(1)$, while some constraints on $\hat{\mathbf{u}}$ must be imposed. For a sub-critical asymptotic family $\langle \tilde{\mathbf{u}}\tilde{\mathbf{a}} \rangle$ appears only in higher than $O(1)$ approximations.

In *Sect.4* and *Appendix A* we obtain the general solution of an asymptotic problem for the critical asymptotic family ($\alpha = 1/2$) with an arbitrary purely oscillating velocity $\tilde{\mathbf{u}}$ and arbitrary initial conditions for $\hat{\mathbf{a}}$. Each solution $\hat{\mathbf{a}} = \bar{\mathbf{a}} + \tilde{\mathbf{a}}$ consists of explicit formulae for its oscillating part $\tilde{\mathbf{a}}$ and an advection-diffusion-type equation for the averaged part $\bar{\mathbf{a}}$. Our detailed analytical calculations include five steps of successive approximations. The obtained solutions satisfy the governing equations with small explicitly calculated residual right-hand-sides. It appears that evolution of the averaged field $\bar{\mathbf{a}}$ includes a drift with the velocity $\bar{\mathbf{V}} = \bar{\mathbf{V}}_0 + \varepsilon \bar{\mathbf{V}}_1 + \varepsilon^2 \bar{\mathbf{V}}_2 + O(\varepsilon^3)$ and pseudo-diffusion with a matrix of coefficients $\bar{\kappa}_{ik} = \varepsilon^2 \bar{\chi}_{ik} + O(\varepsilon^3)$, where $\varepsilon \equiv \varepsilon_{1/2}$ and $\bar{\mathbf{V}}_0, \bar{\mathbf{V}}_1, \bar{\mathbf{V}}_2, \bar{\chi}_{ik}$ have magnitudes $O(1)$. Our term *pseudo-diffusion* means that the evolution of $\bar{\mathbf{a}}$ is described by an advection-diffusion-type equation where the diffusion-type term appears not in the leading approximation.

In *Sect.5* and *Appendix B* we present the asymptotic solutions for supercritical asymptotic families with $\alpha = 1/3$ and $1/4$ and degenerated velocity field $\tilde{\mathbf{u}}$. For $\alpha = 1/3$ the required degeneration is $\bar{\mathbf{V}}_0 \equiv 0$ which leads to $\bar{\mathbf{V}} = \bar{\mathbf{V}}_1 + \varepsilon \bar{\mathbf{V}}_2 + O(\varepsilon^2)$ and $\bar{\kappa}_{ik} \equiv 0 + O(\varepsilon^2)$, where $\varepsilon \equiv \varepsilon_{1/3}$. For $\alpha = 1/4$ the required degeneration is $\bar{\mathbf{V}}_0 \equiv 0$ and $\bar{\mathbf{V}}_1 \equiv 0$ which leads

to $\overline{\mathbf{V}} = \overline{\mathbf{V}}_2 + O(\varepsilon)$ and $\overline{\kappa}_{ik} \equiv 0 + O(\varepsilon)$, where $\varepsilon \equiv \varepsilon_{1/4}$. Hence, the vanishing of the main term of a drift velocity $\overline{\mathbf{V}}_0 \equiv 0$ (or the first two terms $\overline{\mathbf{V}}_0 \equiv 0$ and $\overline{\mathbf{V}}_1 \equiv 0$) does not lead to the smallness of a dimensionless drift. Instead, a different expression $\overline{\mathbf{V}} = O(1)$ appears from a different asymptotic procedure. At the same time any supercritical oscillations do not produce pseudo-diffusion (within the precision of our calculations).

In *Sect.6* we generalise the results of *Sects.4* and *5* considering more general $\widehat{\mathbf{u}}$, which represents a power series of a small parameter ε and contains both oscillating and non-oscillating parts. The expressions for drift velocities appear to be much more general, while pseudo-diffusion stays the same as in *Sect.4*.

Sect.7 contains eleven examples of drift velocities and/or pseudo-diffusivity calculated for various flows. These examples are: (1) the velocity field $\widehat{\mathbf{u}}$ consisting of two arbitrary modulated fields of the same frequency; (2) $\widehat{\mathbf{u}}$ that represents the Fourier series of modulated fields; (3) the Stokes drift, which allows us to make comparison (in *Sect.10*) with classical results; (4) $\widehat{\mathbf{u}}$ is a spherical ‘acoustic’ wave; (5) $\overline{\mathbf{V}}_1$ -drift (*Sect.5*), which takes place for $\overline{\mathbf{V}}_0 \equiv 0$; (6) $\widehat{\mathbf{u}}$ is a plane travelling wave of a general shape or the superposition of two such waves; (7) the polynomial velocity $\widehat{\mathbf{u}}$, which has a nonzero average and represents polynomial of a small parameter; (8) $\widehat{\mathbf{u}}$ as the Bjorknes configuration of two pulsating point sources; (9) a simple $\widehat{\mathbf{u}}$ that produces a drift $\overline{\mathbf{V}}_0$ leading to chaotic dynamics of material particles; (10) the rigid-body-type oscillations of a fluid where a pseudo-diffusion term in the equations appears naturally; this example explains both mathematical and physical nature of pseudo-diffusion; (11) we consider a sub-critical asymptotic procedure and show that the averaged equations for $\overline{\mathbf{a}}$ contain *pseudo-drift* and pseudo-diffusion; it emphasizes the significance of the critical and super-critical procedures.

The *TTAM* used in *Sects.3-7* combines five components: (i) a version of the two-timing method; (ii) the averaging over fast-time; (iii) the inspection procedure with the multiplicity of asymptotic procedures; (iv) the obtaining of general asymptotic solutions for these procedures; and (v) the estimating of RHS-residuals (which has been only mentioned in *Sects.4-6* and is actually given in *Appendices A,B*).

In *Sect.8* we study the links between the *TTAM*-results of *Sects.2-7* and other theories: (i) the dynamical systems approach, see Aref (1984), Ottino (1989), Samelson & Wiggins (2006)); (ii) *GLM*-theory by Andrews & McIntyre (1978); (iii) the classical drift theory by Stokes (1847), Longuet-Higgins (1953), Batchelor (1967); (iv) the Krylov-Bogoliubov averaging method, see Bogoliubov & Mitropolskii (1961), Krylov & Bogoliubov (1947), Sanders & Verhulst (1985); (v) the homogenization method, see Bensoussan, Lions and Papanicolaou (1978), Berdichevsky, Jikov, and Papanicolaou (1997); (vi) the theory with nonzero molecular diffusion; and (vii) the MHD-kinematics in oscillating flows, see Moffatt (1978). The links between our *TTAM*-results of *Sects.2-7* and the theories (i), (ii) can be established only if we adapt and develop last two theories. Therefore, in *Sect.8.1* we calculate a drift velocity by the averaging of characteristics for transport equations and in *Sect.8.2* we develop a two-timing version of *GLM*-kinematics. It is important that the *TTAM*-theory of *Sects.2-7* uses the Eulerian averaging while the theories (i) and (ii) operate with the Lagrangian averaging. These different averaging operations lead to different results for a drift and pseudo-diffusion.

Finally, in *Sect.9* we discuss the assumptions used, the results obtained, and the problems to be addressed. We also present short discussion/remarks at the end of each section and subsection.

2. Basic Equations and Operations

2.1. Exact problem

The equation for a scalar field $a = a(\mathbf{x}^*, s^*)$ in Cartesian coordinates $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$ and time s^* is

$$\left(\frac{\partial}{\partial s^*} + (\mathbf{u}^* \cdot \nabla^*) \right) a(\mathbf{x}^*, s^*) = 0 \quad (2.1)$$

where $\mathbf{u}^* = \mathbf{u}^*(\mathbf{x}^*, s^*)$ is a *given velocity field*, the asterisks mark dimensional variables. All functions used in this paper are considered to be sufficiently smooth. Equation (2.1) describes the motion of a Lagrangian marker in either an incompressible or compressible fluid or the advection of a passive scalar admixture with concentration $a(\mathbf{x}^*, s^*)$ in an incompressible fluid. For a compressible fluid it also describes the advection of a passive scalar admixture, where \hat{a} represents the ratio of concentration of admixture to density of a fluid. We study the motion of a Lagrangian marker (2.1) due to our primary interest in the motions of material particles.

The velocity field \mathbf{u}^* has the functional form of a *hat-function*

$$\mathbf{u}^* = \hat{\mathbf{u}}^*(\mathbf{x}^*, t^*, \tau); \quad t^* = s^*, \quad \tau = \omega^* s^* \quad (2.2)$$

where t^* and τ are two mutually dependent time variables, which are introduced as two different time-scales $k_1 s^*$ and $k_2 s^*$ with $k_1 = 1$ and $k_2 = \omega^*$ (ω^* is frequency). The τ -dependence in (2.2) is always 2π -periodic. The functional form (2.2) is aimed to describe modulated oscillations of high-frequency. Following established terminology we call t^* *slow time* and τ *fast time*. We do not require (2.2) to satisfy any equations of motion; such a general setting is often accepted, for example in the kinematic MHD-dynamo theory (Moffatt (1978)).

The functional structure of $\hat{\mathbf{u}}^*$ (2.2) underpins the idea that the solution of (2.1) also represents a hat-function:

$$a = \hat{a}(\mathbf{x}^*, t^*, \tau) \quad (2.3)$$

Then after the use of the chain rule the equation (2.1) is

$$\mathfrak{D}^* \hat{a} \equiv \left(\omega^* \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t^*} + (\hat{\mathbf{u}}^* \cdot \nabla^*) \right) \hat{a} = 0 \quad (2.4)$$

$$\frac{\partial}{\partial s^*} = \omega^* \frac{\partial}{\partial \tau} \Big|_{\mathbf{x}^*, t^*} + \frac{\partial}{\partial t^*} \Big|_{\mathbf{x}^*, \tau} \equiv \omega^* \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t^*} \quad (2.5)$$

We also denote partial derivatives by subscripts, *e.g.* $a_\tau \equiv \partial a / \partial \tau$.

We obtain the dimensionless form of (2.4) by introducing the characteristic scales of slow time T , length L , and velocity U . An example of a velocity field is

$$\hat{\mathbf{u}}^* = U(1 + \sin \tau)(1 + C \sin(t^*/T)) \mathbf{g}(x^*/L, y^*/L, y^*/L) \quad (2.6)$$

where \mathbf{g} is an arbitrary vector-function of three scalar variables (one can also take $T = L/U$, but it narrows the class of velocity fields). Dimensionless variables and parameters are not asteriated:

$$t = t^*/T, \quad \mathbf{x} = \mathbf{x}^*/L, \quad \hat{\mathbf{u}} = \hat{\mathbf{u}}^*/U, \quad \omega = \omega^* T \quad (2.7)$$

According to Buckingham's π -theorem the problem (2.4)-(2.7) possesses two independent dimensionless scaling parameters

$$\varepsilon_1 \equiv 1/\omega^* T = 1/\omega, \quad \delta \equiv U/(\omega^* L) \quad (2.8)$$

where δ can be interpreted as either of two physical parameters: (i) characteristic dimensionless displacement $\delta = \delta^*/L$ (for displacement $\delta^* \equiv U/\omega^*$); or (ii) Strouhal number $\delta = \Omega/\omega^*$ (for frequency $\Omega \equiv U/L$). The parameter ε_1 represents another Strouhal number and follows a more general notation:

$$\varepsilon_\alpha \equiv 1/\omega^\alpha, \quad \text{with } \alpha = \text{const} > 0. \quad (2.9)$$

The dimensionless versions of (2.5), (2.4) are

$$\mathfrak{D}\hat{a} \equiv \left(\omega \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} + \omega \delta \hat{\mathbf{u}} \cdot \nabla \right) \hat{a} = 0 \quad (2.10)$$

$$\frac{\partial}{\partial s} = \omega \left. \frac{\partial}{\partial \tau} \right|_{\mathbf{x}, t} + \left. \frac{\partial}{\partial t} \right|_{\mathbf{x}, \tau} \equiv \omega \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} \quad (2.11)$$

Eqn. (2.10) can also be written in the form containing only small parameters:

$$\mathfrak{D}\hat{a}/\omega = \hat{a}_\tau + \varepsilon_1 \hat{a}_t + \delta \hat{\mathbf{u}} \cdot \nabla \hat{a} = 0 \quad (2.12)$$

One can see that we operate on the plane $(\varepsilon_\alpha, \delta)$ of two independent dimensionless scaling parameters. To use a rigorous asymptotic procedure one has to choose a one-parametric asymptotic path on this plane. In order to form such a path one might prescribe the dependence of each characteristic parameter in (2.7) on the chosen single parameter ε_α . The simplest choice is

$$T = \text{const}, \quad L = \text{const}, \quad UT/L = O(\varepsilon_\alpha) \quad (2.13)$$

We will exploit different options for paths, which go to the same limit

$$(\varepsilon_\alpha, \delta(\varepsilon_\alpha)) \rightarrow 0 \quad \text{as } \varepsilon_\alpha \rightarrow 0 \quad (2.14)$$

Remark: More details on the scaling procedure are given in *Sect.2.3* and *Sect.7.3*.

2.2. The classes of \mathbb{H} , \mathbb{B} , \mathbb{T} , and $\mathbb{O}(1)$ -functions

Notations and definitions:

Definition 1. The class \mathbb{H} of *hat-functions* is defined as

$$\hat{f} \in \mathbb{H}: \quad \hat{f}(\mathbf{x}, t, \tau) = \hat{f}(\mathbf{x}, t, \tau + 2\pi) \quad (2.15)$$

where $t = s$ and $\tau \equiv \omega s$ are two mutually dependent time variables (ω^* is dimensionless frequency); the τ -dependence is always 2π -periodic; the dependencies on \mathbf{x} and t are not specified.

Comments: (A) We have already accepted that $\mathbf{u}^* \in \mathbb{H}$ (2.2) and $a \in \mathbb{H}$ (2.3). (B) For any $\hat{f} \in \mathbb{H}$ a partial time-derivative can be expressed *via* the chain rule as

$$\frac{\partial \hat{f}}{\partial s} = \left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \tau} \right) \hat{f}(\mathbf{x}, t, \tau) \equiv \hat{f}_t + \omega \hat{f}_\tau, \quad \text{where } t = s, \tau = \omega s \quad (2.16)$$

(C) In any version of the two-timing method the variables t and τ are considered to be mutually independent after the chain rule (2.16) has been applied. The justification of this auxiliary assumption is given *a posteriori* (see *Appendix A*).

Definition 2. For an arbitrary $\hat{f} \in \mathbb{H}$ the *Eulerian averaging operation* is

$$\langle \hat{f} \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + 2\pi} \hat{f}(\mathbf{x}, t, \tau) d\tau, \quad \forall \tau_0 \quad (2.17)$$

where during the integration $t = \text{const}$ and $\langle \hat{f} \rangle$ does not depend on τ_0 .

Definition 3. The class \mathbb{T} of *tilde-functions* is such that

$$\tilde{f} \in \mathbb{T}: \quad \tilde{f}(\mathbf{x}, t, \tau) = \tilde{f}(\mathbf{x}, t, \tau + 2\pi), \quad \text{with} \quad \langle \tilde{f} \rangle = 0, \quad (2.18)$$

Comments: (A) One can see that a \mathbb{T} -function represents a special case of \mathbb{H} -function (2.15) with zero average (2.17). (B) In the text the tilde-functions are also called the purely oscillating functions.

Definition 4. The class \mathbb{B} of *bar-functions* is defined as

$$\bar{f} \in \mathbb{B}: \quad \bar{f}_\tau \equiv 0, \quad \bar{f}(\mathbf{x}, t) = \langle \bar{f}(\mathbf{x}, t) \rangle \quad (2.19)$$

Comments: (A) Any \mathbb{B} -function depends on \mathbf{x} and t and does not depend on τ ; in particular, \mathbb{B} -function can appear from \mathbb{H} -function after its averaging. (B) With the use of the average (2.17) any \mathbb{H} -function can be uniquely separated into its \mathbb{B} - and \mathbb{T} - parts:

$$\hat{f}(\mathbf{x}, t, \tau) = \bar{f}(\mathbf{x}, t) + \tilde{f}(\mathbf{x}, t, \tau) \quad \text{with} \quad \langle \hat{f}(\mathbf{x}, t, \tau) \rangle \equiv \bar{f}(\mathbf{x}, t) \quad (2.20)$$

Definition 5. The \mathbb{T} -integration: For a given \tilde{f} we introduce a new function \tilde{f}^τ called the \mathbb{T} -integral of \tilde{f} :

$$\tilde{f}^\tau \equiv \int_0^\tau \tilde{f}(\mathbf{x}, t, \sigma) d\sigma - \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\mu \tilde{f}(\mathbf{x}, t, \sigma) d\sigma \right) d\mu. \quad (2.21)$$

which represents the unique solution of the partial differential equation $\partial \tilde{f}^\tau / \partial \tau = \tilde{f}$ with the condition $\langle \tilde{f} \rangle = \langle \tilde{f}^\tau \rangle = 0$ (2.18).

Comments: (A) The τ -derivative of \mathbb{T} -function always represents \mathbb{T} -function. However the τ -integration of \mathbb{T} -function can produce an \mathbb{H} -function. An example: $\tilde{f} = \bar{f}_1 \sin \tau$ where \bar{f}_1 is an arbitrary function of \mathbf{x}, t : one can see that $\langle \tilde{f} \rangle \equiv 0$, however $\langle \int_0^\mu \tilde{f} d\tau \rangle = \bar{f}_1 \neq 0$, unless $\bar{f}_1 \equiv 0$. Formula (2.21) keeps the result of integration inside the \mathbb{T} -class. (B) The \mathbb{T} -integration is inverse to the τ -differentiation

$$(\tilde{f}^\tau)_\tau = (\tilde{f}_\tau)^\tau = \tilde{f}. \quad (2.22)$$

The proof is omitted.

Definition 6. A dimensionless function $f = f(\mathbf{x}, t, \tau, \varepsilon)$ (ε is a small parameter) belongs to the class $\mathbb{O}(1)$ if $f = O(1)$ and all partial \mathbf{x} -, t -, and τ -derivatives of f (required for further consideration) are also $O(1)$.

Comments: In all text below: (A) The highest spatial derivatives will be of the fourth order, while all t - and τ -derivatives will be of the first order; (B) All large or small parameters (in all this paper) are represented by various degrees of ω only; these parameters appear as explicit multipliers in all formulae containing tilde- and bar-functions; these functions always belong to $\mathbb{O}(1)$ -class.

Definition 7. The commutator of two vector fields $\mathbf{a}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ is

$$[\mathbf{a}, \mathbf{b}] \equiv (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b}. \quad (2.23)$$

Comment: The commutator is antisymmetric and always satisfies Jacobi's identity for three vector fields $\mathbf{a}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$, and $\mathbf{c}(\mathbf{x})$:

$$[\mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}], \quad [\mathbf{a}, [\mathbf{b}, \mathbf{c}]] + [\mathbf{c}, [\mathbf{a}, \mathbf{b}]] + [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] = 0 \quad (2.24)$$

Two more useful properties of commutators are:

$$\operatorname{div} \mathbf{a} = 0, \quad \operatorname{div} \mathbf{b} = 0 \quad \Rightarrow \quad \operatorname{div} [\mathbf{a}, \mathbf{b}] = 0 \quad (2.25)$$

$$\mathbf{a} \cdot \mathbf{n} = 0, \quad \mathbf{b} \cdot \mathbf{n} = 0 \quad \Rightarrow \quad [\mathbf{a}, \mathbf{b}] \cdot \mathbf{n} = 0 \quad (2.26)$$

where \mathbf{n} is a normal vector to a surface $\partial\Omega$.

Properties of τ -differentiation and \mathbb{T} -integration:

For τ -derivatives it is clear that

$$\widehat{f}_\tau = \overline{f}_\tau + \widetilde{f}_\tau = \widetilde{f}_\tau, \quad \langle \widehat{f}_\tau \rangle = \langle \widetilde{f}_\tau \rangle = 0 \quad (2.27)$$

The product of two \mathbb{T} -functions \widetilde{f} and \widetilde{g} forms a \mathbb{H} -function: $\widetilde{f}\widetilde{g} \equiv \widehat{F}$, say. Separating \mathbb{T} -part \widetilde{F} from \widehat{F} we write

$$\widetilde{F} = \widehat{F} - \langle \widehat{F} \rangle = \widetilde{f}\widetilde{g} - \langle \widetilde{f}\widetilde{g} \rangle = \widetilde{\widetilde{f}\widetilde{g}} \equiv \{\widetilde{f}\widetilde{g}\} \quad (2.28)$$

Since we do not use a two-level tilde notation for the \mathbb{T} -parts of long expressions, we introduce braces instead. As the average operation (2.17) is proportional to the integration over τ , for products containing functions $\widetilde{f}, \widetilde{g}, \widetilde{h}$ and their derivatives we have

$$\langle \widetilde{f}\widetilde{g}_\tau \rangle = \langle (\widetilde{f}\widetilde{g})_\tau \rangle - \langle \widetilde{f}_\tau \widetilde{g} \rangle = -\langle \widetilde{f}_\tau \widetilde{g} \rangle = -\langle \widetilde{f}_\tau \widehat{g} \rangle \quad (2.29)$$

$$\langle \widetilde{f}_\tau \widetilde{g} \widetilde{h} \rangle + \langle \widetilde{f} \widetilde{g}_\tau \widetilde{h} \rangle + \langle \widetilde{f} \widetilde{g} \widetilde{h}_\tau \rangle = 0 \quad (2.30)$$

$$\langle \widetilde{f}\widetilde{g}^\tau \rangle = \langle (\widetilde{f}^\tau \widetilde{g}^\tau)_\tau \rangle - \langle \widetilde{f}^\tau \widetilde{g} \rangle = -\langle \widetilde{f}^\tau \widetilde{g} \rangle = -\langle \widetilde{f}^\tau \widehat{g} \rangle \quad (2.31)$$

which represent different versions of the integration by parts. Similarly

$$\langle [\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}_\tau] \rangle = -\langle [\widetilde{\mathbf{a}}_\tau, \widetilde{\mathbf{b}}] \rangle = -\langle [\widetilde{\mathbf{a}}_\tau, \widehat{\mathbf{b}}] \rangle, \quad \langle [\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}^\tau] \rangle = -\langle [\widetilde{\mathbf{a}}^\tau, \widetilde{\mathbf{b}}] \rangle = -\langle [\widetilde{\mathbf{a}}^\tau, \widehat{\mathbf{b}}] \rangle \quad (2.32)$$

The equation to be solved:

Let us accept that a dimensionless velocity in (2.10), (2.12) is

$$\widehat{\mathbf{u}}(\mathbf{x}, t, \tau) \in \mathbb{H} \cap \mathbb{O}(1), \quad \text{and} \quad \delta(\omega) = \delta_0 \omega^{\beta-1}, \quad \beta = \text{const} < 1 \quad (2.33)$$

where $\widehat{\mathbf{u}}$ is a given function; indefinite constant β allows us to consider different asymptotic paths; one can take $\delta_0 \equiv 1$ since any other δ_0 can be incorporated into $\widehat{\mathbf{u}}$. Eqn. (2.10) can be rewritten as

$$\mathfrak{D}\widehat{\mathbf{a}} = \omega \widehat{\mathbf{a}}_\tau + \widehat{\mathbf{a}}_t + \omega^\beta (\widehat{\mathbf{u}} \cdot \nabla) \widehat{\mathbf{a}} = 0. \quad (2.34)$$

In Sects.3-7 we will study and solve this equation.

Remark: The τ -periodicity of the given velocity (2.2),(2.33) represents a restriction which is artificially imposed to simplify further calculations. At the same time we will show that a τ -periodic asymptotic solution (2.3) can be build for any initial data; hence we are not forced to go outside the \mathbb{H} -class (2.15).

2.3. Choice of scale T for t -independent velocity

A t -independent velocity

$$\widehat{\mathbf{u}} = \widehat{\mathbf{u}}(\mathbf{x}, \tau) \quad (2.35)$$

is important theoretically. This case represents a degeneration, where the presence of slow time t in eqns.(2.10),(2.34) is provided only by the additional agreement $\widehat{\mathbf{a}}_t \in \mathbb{O}(1)$ leading to two mutually dependent time-scales $(t, \tau) = (s, \omega s)$. It is clear that for (2.35) this agreement represents only one available option and an infinite number of different slow-time-scales can be introduced. To make it possible one can write

$$\tau = \omega^{1-\lambda} (\omega^\lambda s) = \omega_\lambda t_\lambda \equiv \tau_\lambda \quad \text{where} \quad \omega_\lambda \equiv \omega^{1-\lambda}, \quad t_\lambda \equiv \omega^\lambda s, \quad \lambda = \text{const} < 1 - \lambda_0 \quad (2.36)$$

with a small constant $\lambda_0 > 0$ which provides $\omega_\lambda \rightarrow \infty$ when $\omega \rightarrow \infty$. These manipulations allow to introduce time-scales $(t_\lambda, \tau_\lambda) = (\omega^\lambda s, \omega s)$ and a new version of eqn.(2.34)

$$\left(\frac{\partial}{\partial s} + \omega^\beta \widehat{\mathbf{u}}_\lambda \cdot \nabla \right) \widehat{\mathbf{a}} = 0, \quad \frac{\partial}{\partial s} = \omega^\lambda \frac{\partial}{\partial t_\lambda} + \omega \frac{\partial}{\partial \tau_\lambda} \quad (2.37)$$

with $\hat{\mathbf{u}}_\lambda \equiv \hat{\mathbf{u}}(\mathbf{x}, \tau_\lambda)$. Further transformations of (2.37) yield

$$\left(\frac{\partial}{\partial t_\lambda} + \omega_\lambda \frac{\partial}{\partial \tau_\lambda} \right) \hat{a} + \omega_\lambda^{\beta_\lambda} (\hat{\mathbf{u}}_\lambda \cdot \nabla) \hat{a} = 0, \quad (2.38)$$

where $\beta_\lambda \equiv (\beta - \lambda)/(1 - \lambda)$ and $\hat{a} = \hat{a}(\mathbf{x}, t_\lambda, \tau_\lambda)$. Finally, we replace the old scaling agreement $\hat{a}_t \in \mathbb{O}(1)$ with a new one $\hat{a}_{t_\lambda} \in \mathbb{O}(1)$ ($\hat{a}_{\tau_\lambda} \in \mathbb{O}(1)$ remains valid since $\tau_\lambda = \tau$). Now one can see that the problem (2.38) is mathematically identical to (2.34), which means that for any solution in variables (t, τ) we have an additional solution in variables $(t_\lambda, \tau_\lambda)$ (and *vice versa*); the varying of λ gives us an infinite number of such additional solutions. Hence, an infinite number of new solutions to (2.38) can be obtained by rescaling of the known solutions to (2.34), which will be considered in *Sects.3-7*. The existence of such additional solutions means that motions with any slow-time-scale $\omega^\lambda t$ (2.36) are possible. Indeed, it sounds physically reasonable: if a slow-time-scale is not enforced externally (does not appear in the given velocity) then any time-scale that is longer than imposed oscillations can be considered as a slow one. For brevity, in the text below we keep $\hat{a}_t \in \mathbb{O}(1)$ and two time-scales $(t, \tau) = (s, \omega s)$.

Remark: Similar to (2.35), (2.38), the multiplicity of slow-time-scales takes place, for instance, in dynamics of a classical pendulum with a harmonically vibrating pivot, when an external gravity field is absent.

3. The Inspection Procedure

For simplicity we consider now eqns.(2.33),(2.34) with purely oscillatory velocity $\hat{\mathbf{u}} = \tilde{\mathbf{u}} \in \mathbb{T} \cap \mathbb{O}(1)$. In our *inspection procedure* we use a test solution to (2.34)

$$\hat{a}(\mathbf{x}, t, \tau) = \bar{a}(\mathbf{x}, t) + \frac{1}{\omega^\alpha} \tilde{b}(\mathbf{x}, t, \tau); \quad \alpha = \text{const} > 0 \quad (3.1)$$

where $\bar{a} \in \mathbb{B} \cap \mathbb{O}(1)$, $\tilde{b} \in \mathbb{T} \cap \mathbb{O}(1)$ and the amplitude of an oscillating part is given by the small parameter $\varepsilon_\alpha = 1/\omega^\alpha$ (2.9). The substitution of (3.1) into (2.34) yields

$$\bar{a}_t + \frac{1}{\omega^\alpha} (\omega \tilde{b}_\tau + \tilde{b}_t) + \omega^\beta \left((\tilde{\mathbf{u}} \cdot \nabla) \bar{a} + \frac{1}{\omega^\alpha} (\tilde{\mathbf{u}} \cdot \nabla) \tilde{b} \right) = 0 \quad (3.2)$$

The \mathbb{B} -part of (3.2) is

$$\bar{a}_t + \omega^{\beta-\alpha} \langle (\tilde{\mathbf{u}} \cdot \nabla) \tilde{b} \rangle = 0 \quad (3.3)$$

and the subtraction of (3.3) from (3.2) produces the \mathbb{T} -part of (3.2)

$$\underline{\omega^{1-\alpha} \tilde{b}_\tau + \omega^{-\alpha} \tilde{b}_t} + \underline{\omega^\beta (\tilde{\mathbf{u}} \cdot \nabla) \bar{a}} + \omega^{\beta-\alpha} \{ (\tilde{\mathbf{u}} \cdot \nabla) \tilde{b} \} = 0 \quad (3.4)$$

with the notation $\{\cdot\}$ (2.28) used. In each underlined group in (3.4) the first term dominates over the second one (due to $\alpha > 0$ (3.1)). Therefore, to make this equation meaningful we need to accept that the dominating terms are of the same order:

$$1 - \alpha = \beta \quad \text{or} \quad \alpha + \beta = 1. \quad (3.5)$$

Then the imposed restrictions $\alpha > 0$ (3.1) and $\beta < 1$ (2.33) are mutually compatible and (3.3), (3.4) can be rewritten as

$$\bar{a}_t + \omega^\gamma \langle (\tilde{\mathbf{u}} \cdot \nabla) \tilde{b} \rangle = 0 \quad \text{where} \quad \gamma \equiv 2\beta - 1 = 1 - 2\alpha \quad (3.6)$$

$$\omega \left(\tilde{b}_\tau + (\tilde{\mathbf{u}} \cdot \nabla) \bar{a} \right) + \tilde{b}_t + \omega^\beta \{ (\tilde{\mathbf{u}} \cdot \nabla) \tilde{b} \} = 0 \quad (3.7)$$

The second term in eqn.(3.6) represents the changes of \bar{a} due to the averaged nonlinear effects of oscillations. Taking the different values of γ we introduce three qualitatively different asymptotic families of solutions:

$$\gamma > 0 \ (1/2 < \beta < 1, \ 0 < \alpha < 1/2) \quad -\text{super-critical oscillations} \quad (3.8)$$

$$\gamma = 0 \ (\beta = 1/2, \ \alpha = 1/2) \quad -\text{critical oscillations} \quad (3.9)$$

$$\gamma < 0 \ (-\infty < \beta < 1/2, \ 1/2 < \alpha < \infty) \quad -\text{sub-critical oscillations} \quad (3.10)$$

One can notice that *super-critical*, *critical*, and *sub-critical oscillations* have been introduced as the asymptotic families of solutions, not as particular solutions (recall that any particular solution can be treated as a member of several different asymptotic families).

For the family of *critical oscillations* the averaged oscillatory term in (3.6) has the same order $O(1)$ as a mean solution. Then the leading terms (for large ω) in (3.6), (3.7) are:

$$\bar{a}_t + \langle (\tilde{\mathbf{u}} \cdot \nabla) \tilde{b} \rangle = 0; \quad (3.11)$$

$$\tilde{b}_\tau + (\tilde{\mathbf{u}} \cdot \nabla) \bar{a} = 0. \quad (3.12)$$

This system gives a single equation for \bar{a}

$$\bar{a}_t - \langle (\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle \bar{a} = 0, \quad \tilde{\xi} \equiv \tilde{\mathbf{u}}^\tau \quad (3.13)$$

which is obtained by \mathbb{T} -integration (2.21) of (3.12) and the following substitution of $\tilde{b}(\mathbf{x}, t, \tau) = -(\tilde{\xi} \cdot \nabla) \bar{a}$ into (3.11).

The critical family of oscillations (3.9) separates the larger families of sub-critical and super-critical oscillations from each other. For super-critical oscillations $\gamma > 0$ and the correlation in (3.6) can not be balanced by $\bar{a}_t = O(1)$. Then the condition of solvability requires this correlation to vanish, hence the leading order of (3.11), (3.13) is replaced with $\langle (\tilde{\mathbf{u}} \cdot \nabla) \tilde{b} \rangle = 0$. It means that the imposed velocity must degenerate (satisfy certain additional restrictions).

Remarks:

1. The inspection procedure is aimed to find and classify all possible asymptotic solutions (of the functional form (3.1)) of a given PDE. This procedure also allows to choose a natural small parameter (2.9) with different values of α for different classes of solutions.

2. The critical and super-critical oscillations are most interesting both theoretically and practically. We justify this statement with three arguments: (A) *Sects.4-7* below are completely devoted to these two oscillatory families. The obtained results are mathematically satisfactory and look physically convincing. (B) These families of oscillations are the least studied, probably due to the reason that for them $U \rightarrow \infty$ as $\omega \rightarrow \infty$ in (2.13); however it is well-known that such an increase of U does not diminish the validity of asymptotic procedures. (C) In *Sect.7.11* we will consider the sub-critical family (3.10) with $\beta = 0$ in (2.33),(2.34). We will show that for this family a drift is replaced with a *pseudo-drift* and the general solution is diverging.

4. Family of Critical Oscillations. Purely Oscillatory Velocity.

4.1. Critical asymptotic procedure with $\alpha = 1/2$

Following (3.9) we accept that (2.33) belongs to a critical asymptotic family:

$$\hat{\mathbf{u}}(\mathbf{x}, t, \tau) = \tilde{\mathbf{u}}(\mathbf{x}, t, \tau), \quad \delta = \omega^{-1/2}, \quad (4.1)$$

Then (2.34), (4.1) yield:

$$\mathfrak{D}\hat{a} = \omega\hat{a}_\tau + \hat{a}_t + \sqrt{\omega}(\tilde{\mathbf{u}} \cdot \nabla)\hat{a} = 0 \quad (4.2)$$

The small parameter $\varepsilon_{1/2} = 1/\sqrt{\omega}$ (2.9) allows to rewrite it as

$$\mathfrak{D}_2\hat{a} \equiv \hat{a}_\tau + \varepsilon(\tilde{\mathbf{u}} \cdot \nabla)\hat{a} + \varepsilon^2\hat{a}_t = 0, \quad \mathfrak{D}_2 \equiv \varepsilon^2\mathfrak{D} = \mathfrak{D}/\omega \quad (4.3)$$

where the subscript in $\varepsilon_{1/2}$ has been dropped. We are looking for the solution of (4.3) in the form of regular series

$$\hat{a}(\mathbf{x}, t, \tau) = \sum_{k=0}^{\infty} \varepsilon^k \hat{a}_k(\mathbf{x}, t, \tau), \quad \hat{a}_k \in \mathbb{H} \cap \mathbb{O}(1), \quad k = 0, 1, 2, \dots \quad (4.4)$$

The substitution of (4.4) into (4.3) produces the equations of successive approximations

$$\hat{a}_{0\tau} = 0 \quad (4.5)$$

$$\hat{a}_{1\tau} = -(\tilde{\mathbf{u}} \cdot \nabla)\hat{a}_0 \quad (4.6)$$

$$\hat{a}_{n\tau} = -(\tilde{\mathbf{u}} \cdot \nabla)\hat{a}_{n-1} - \partial_t \hat{a}_{n-2}, \quad \partial_t \equiv \partial/\partial t, \quad n = 2, 3, 4, \dots \quad (4.7)$$

$$\hat{a}_k = \bar{a}_k(\mathbf{x}, t) + \tilde{a}_k(\mathbf{x}, t, \tau), \quad \bar{a}_k \in \mathbb{B} \cap \mathbb{O}(1), \quad \tilde{a}_k \in \mathbb{T} \cap \mathbb{O}(1), \quad k = 0, 1, 2, \dots$$

where (4.6) represents a systematically derived version of (3.12). Let $\hat{a}^{[n]}$ be a truncated solution (4.4) of order n :

$$\hat{a}^{[n]}(\mathbf{x}, t, \tau) \equiv \sum_{k=0}^n \varepsilon^k \hat{a}_k(\mathbf{x}, t, \tau) \quad (4.8)$$

Its substitution into (4.3) produces the RHS-residual $\text{Res}[n]$:

$$\mathfrak{D}_2\hat{a}^{[n]} = \text{Res}[n] \quad (4.9)$$

4.2. General solution for the first five approximations

The detailed solving of (4.5)-(4.7) for $n = 0, 1, 2, 3, 4$ is given in *Appendix A*. The truncated general solution $\hat{a}^{[4]}$ is

$$\hat{a}^{[4]} = \hat{a}_0 + \varepsilon\hat{a}_1 + \varepsilon^2\hat{a}_2 + \varepsilon^3\hat{a}_3 + \varepsilon^4\hat{a}_4 \quad (4.10)$$

with the tilde-parts \tilde{a}_k given by explicit recurrent expressions

$$\tilde{a}_0 \equiv 0, \quad (4.11)$$

$$\tilde{a}_1 = -(\tilde{\xi} \cdot \nabla)\bar{a}_0, \quad (4.12)$$

$$\tilde{a}_2 = -(\tilde{\xi} \cdot \nabla)\bar{a}_1 - \{(\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_1\}^\tau, \quad (4.13)$$

$$\tilde{a}_3 = -(\tilde{\xi} \cdot \nabla)\bar{a}_2 - \{(\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_2\}^\tau - \tilde{a}_{1t}^\tau, \quad (4.14)$$

$$\tilde{a}_4 = -(\tilde{\xi} \cdot \nabla)\bar{a}_3 - \{(\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_3\}^\tau - \tilde{a}_{2t}^\tau, \quad (4.15)$$

$$\tilde{\xi} \equiv \tilde{\mathbf{u}}^\tau = \int_0^\tau \tilde{\mathbf{u}}(\mathbf{x}, t, \sigma) d\sigma - \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\mu \tilde{\mathbf{u}}(\mathbf{x}, t, \sigma) d\sigma \right) d\mu \quad (4.16)$$

and the bar-parts \bar{a}_k satisfy the equations

$$(\partial_t + \bar{\mathbf{V}}_0 \cdot \nabla) \bar{a}_0 = 0 \quad (4.17)$$

$$(\partial_t + \bar{\mathbf{V}}_0 \cdot \nabla) \bar{a}_1 + (\bar{\mathbf{V}}_1 \cdot \nabla) \bar{a}_0 = 0 \quad (4.18)$$

$$(\partial_t + \bar{\mathbf{V}}_0 \cdot \nabla) \bar{a}_2 + (\bar{\mathbf{V}}_1 \cdot \nabla) \bar{a}_1 + (\bar{\mathbf{V}}_2 \cdot \nabla) \bar{a}_0 = \frac{\partial}{\partial x_i} \left(\bar{\chi}_{ik} \frac{\partial \bar{a}_0}{\partial x_k} \right), \quad (4.19)$$

$$\hat{\mathbf{V}}_0 \equiv \frac{1}{2}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}], \quad \bar{\mathbf{V}}_1 \equiv \frac{1}{3}\langle [[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}], \tilde{\boldsymbol{\xi}}] \rangle, \quad (4.20)$$

$$\bar{\mathbf{V}}_2 \equiv \frac{1}{4}\langle [[\hat{\mathbf{V}}_0, \tilde{\boldsymbol{\xi}}], \tilde{\boldsymbol{\xi}}] \rangle + \frac{1}{2}\langle [\tilde{\mathbf{V}}_0, \tilde{\mathbf{V}}_0^\tau] \rangle + \frac{1}{2}\langle [\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}_t] \rangle + \frac{1}{2}\langle \tilde{\boldsymbol{\xi}} \operatorname{div} \tilde{\mathbf{u}}' + \tilde{\mathbf{u}}' \operatorname{div} \tilde{\boldsymbol{\xi}} \rangle, \quad (4.21)$$

$$\tilde{\mathbf{u}}' \equiv \tilde{\boldsymbol{\xi}}_t - [\bar{\mathbf{V}}_0, \tilde{\boldsymbol{\xi}}], \quad (4.22)$$

$$2\bar{\chi}_{ik} \equiv \langle \tilde{u}'_i \tilde{\xi}_k + \tilde{u}'_k \tilde{\xi}_i \rangle = \mathfrak{L}_{\bar{\mathbf{V}}_0} \langle \tilde{\xi}_i \tilde{\xi}_k \rangle, \quad (4.23)$$

$$\mathfrak{L}_{\bar{\mathbf{V}}_0} \bar{f}_{ik} \equiv (\partial_t + \bar{\mathbf{V}}_0 \cdot \nabla) \bar{f}_{ik} - \frac{\partial \bar{V}_{0k}}{\partial x_m} \bar{f}_{im} - \frac{\partial \bar{V}_{0i}}{\partial x_m} \bar{f}_{km} \quad (4.24)$$

where the operator $\mathfrak{L}_{\bar{\mathbf{V}}_0}$ is such that $\mathfrak{L}_{\bar{\mathbf{V}}_0} \bar{f}_{ik} = 0$ represents the condition for tensorial field $\bar{f}_{ik}(\mathbf{x}, t)$ to be ‘frozen’ into velocity field $\bar{\mathbf{V}}_0(\mathbf{x}, t)$. Three equations (4.17)-(4.19) can be written as a single advection-pseudo-diffusion equation (with the error $O(\varepsilon^3)$)

$$(\partial_t + \bar{\mathbf{V}} \cdot \nabla) \bar{a} = \frac{\partial}{\partial x_i} \left(\bar{\kappa}_{ik} \frac{\partial \bar{a}}{\partial x_k} \right) \quad (4.25)$$

$$\bar{\mathbf{V}} = \bar{\mathbf{V}}^{[2]} = \bar{\mathbf{V}}_0 + \varepsilon \bar{\mathbf{V}}_1 + \varepsilon^2 \bar{\mathbf{V}}_2, \quad (4.26)$$

$$\bar{\kappa}_{ik} = \bar{\chi}_{ik}^{[2]} = \varepsilon^2 \bar{\chi}_{ik} \quad (4.27)$$

$$\bar{a} = \bar{a}^{[2]} = \bar{a}_0 + \varepsilon \bar{a}_1 + \varepsilon^2 \bar{a}_2 \quad (4.28)$$

where all coefficients are given in (4.20)-(4.24). Eqn. (4.25) shows that the averaged motion of \hat{a} represents a drift with velocity $\bar{\mathbf{V}} = \bar{\mathbf{V}}^{[2]} + O(\varepsilon^3)$ and *pseudo-diffusion* with the matrix coefficient $\bar{\kappa}_{ik} = \varepsilon^2 \bar{\chi}_{ik} + O(\varepsilon^3)$.

Definition: The term *pseudo-diffusion* (PD) means that: (i) the evolution of \bar{a} is described by an advection-diffusion-type equation (4.25); (ii) the diffusion-type term appears in (4.25) not in the leading approximation (see (4.19), where it represents a known source-type-term in the second approximation); and (ii) the equation (4.25) is valid only for regular asymptotic expansions (4.26)-(4.28).

In $\hat{a}^{[4]}$ (4.10) the expressions for \bar{a}_0 , \bar{a}_1 , \bar{a}_2 , \bar{a}_3 , \bar{a}_4 , and \bar{a}_5 are given by (4.11)-(4.24), while \bar{a}_3 and \bar{a}_4 are to be found from higher approximations. The substitution of $\hat{a}^{[4]}$ into (4.3) produces the RHS-residual of order ε^5

$$\mathfrak{D}_2 \hat{a}^{[4]} = \operatorname{Res}[4] = O(\varepsilon^5) = O(1/\omega^{5/2}) \quad (4.29)$$

One can notice that the residual of order $O(1/\omega^{5/2})$ for \mathfrak{D}_2 leads to the residual $O(1/\omega^{3/2})$ for the original operator $\mathfrak{D}(4.2)$.

Remarks:

1. The statement that (4.11)-(4.24) describe the solution of the problem with arbitrary initial data is based on the fact that initial data for (4.17)-(4.19) can be chosen arbitrarily.
2. We have explicitly solved the equations of the first five approximations. Such persistence does not occur often in the use of asymptotic methods. However, the well-known examples are the surface gravity wave, see Stokes (1847), Stoker (1957), Debnath (1994) and the low-*Re*-solutions for a sphere, see Chester, Breach, and Proudmen (1969). Our motivation for doing the cumbersome calculations of *Appendix A* is based on two our

results: (i) the first and the second approximations of the critical asymptotic solutions appear as the main-order terms in super-critical solutions (see *Sect.5*); and (ii) *PD* (which qualitatively complements a drift) appears only in the second approximation.

3. The term *PD* is different from *super-diffusion*, which means the spreading of an admixture faster than \sqrt{t} (see Avallaneda & Majda (1992), Volpert, Nec, and Nepomnyashchy (2010)).

5. Super-critical Oscillations

5.1. Super-critical asymptotic procedure with $\alpha = 1/3$

For critical oscillations (4.1) the drift velocity (4.26) is $\bar{\mathbf{V}} \sim \bar{\mathbf{V}}_0 = O(1)$ (4.20). From (4.26) one can expect that for the degenerated $\tilde{\mathbf{u}}$ such that

$$\bar{\mathbf{V}}_0 = \frac{1}{2} \langle [\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}] \rangle \equiv 0 \quad (5.1)$$

the drift is $\bar{\mathbf{V}} \sim \varepsilon \bar{\mathbf{V}}_1 = O(1/\sqrt{\omega})$. However, one can find that the drift velocity is still $O(1)$ in a different asymptotic family. To show it, we accept (5.1) and consider a super-critical family of purely oscillating velocities

$$\hat{\mathbf{u}}(\mathbf{x}, t, \tau) = \tilde{\mathbf{u}}(\mathbf{x}, t, \tau), \quad \delta = \omega^{-1/3} \quad (5.2)$$

where $\tilde{\mathbf{u}}(\mathbf{x}, t, \tau) \in \mathbb{T} \cap \mathbb{O}(1)$ is a given function. Then eqns. (2.34), (5.2) yield:

$$\mathfrak{D}\hat{a} = \omega \hat{a}_\tau + \hat{a}_t + \omega^{2/3}(\tilde{\mathbf{u}} \cdot \nabla) \hat{a} = 0 \quad (5.3)$$

A small parameter $\varepsilon_{1/3} = \omega^{-1/3}$ allows to rewrite (5.3) as

$$\mathfrak{D}_3 \hat{a} \equiv \hat{a}_\tau + \varepsilon(\tilde{\mathbf{u}} \cdot \nabla) \hat{a} + \varepsilon^3 \hat{a}_t = 0, \quad \mathfrak{D}_3 \equiv \varepsilon^3 \mathfrak{D} = \mathfrak{D}/\omega \quad (5.4)$$

where the subscript in $\varepsilon_{1/3}$ is dropped. We are looking for a solution of (5.4) in the form of regular series (4.4) with the redefined ε . The substitution of (4.4) into (5.4) produces the equations for successive approximations (*cf.* with (4.5)-(4.7))

$$\hat{a}_{0\tau} = 0 \quad (5.5)$$

$$\hat{a}_{1\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_0 \quad (5.6)$$

$$\hat{a}_{2\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_1 \quad (5.7)$$

$$\hat{a}_{n\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_{n-1} - \partial_t \hat{a}_{n-3}, \quad n = 3, 4, 5, \dots \quad (5.8)$$

These equations can be solved in the most general form (see *Appendix B*). Here we present the truncated solution $\hat{a}^{[4]}$ (4.10). For the tilde-part of each approximation one can obtain explicit recurrent formulae

$$\tilde{a}_0 \equiv 0, \quad (5.9)$$

$$\tilde{a}_1 = -(\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_0, \quad (5.10)$$

$$\tilde{a}_2 = -(\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_1 - \{(\tilde{\mathbf{u}} \cdot \nabla) \tilde{a}_1\}^\tau \quad (5.11)$$

$$\tilde{a}_3 = -(\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_2 - \{(\tilde{\mathbf{u}} \cdot \nabla) \tilde{a}_2\}^\tau \quad (5.12)$$

$$\tilde{a}_4 = -(\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_3 - \{(\tilde{\mathbf{u}} \cdot \nabla) \tilde{a}_3\}^\tau - \tilde{a}_{1t}^\tau, \forall \bar{a}_3 \quad (5.13)$$

and the transport equation (which is valid with the error $O(\varepsilon^3)$) for averaged motion

$$(\partial_t + \overline{\mathbf{V}} \cdot \nabla) \overline{a} = 0, \quad (5.14)$$

$$\overline{\mathbf{V}} = \overline{\mathbf{V}}^{[2]} = \overline{\mathbf{V}}_1 + \varepsilon \overline{\mathbf{V}}_2, \quad (5.15)$$

$$\overline{a} = \overline{a}^{[2]} = \overline{a}_0 + \varepsilon \overline{a}_1, \quad \forall \overline{a}_2, \overline{a}_3, \overline{a}_4 \quad (5.16)$$

$$\overline{\mathbf{V}}_1 = -\frac{1}{3} \langle [\widehat{\mathbf{K}}, \widetilde{\boldsymbol{\xi}}] \rangle, \quad \overline{\mathbf{V}}_2 = -\frac{1}{8} (\overline{\boldsymbol{\kappa}} + \overline{\mathbf{K}''}), \quad (5.17)$$

$$\widehat{\mathbf{K}} \equiv [\widetilde{\boldsymbol{\xi}}, \widetilde{\mathbf{u}}], \quad \widehat{\boldsymbol{\kappa}} \equiv [\widetilde{\mathbf{K}}^\tau, \widetilde{\mathbf{K}}], \quad \widehat{\mathbf{K}}'' \equiv [[\widehat{\mathbf{K}}, \widetilde{\boldsymbol{\xi}}], \widetilde{\boldsymbol{\xi}}] \quad (5.18)$$

Eqn.(5.14) means that the averaged motion of a passive admixture (up to the first two approximations) represents a pure drift with velocity $\overline{\mathbf{V}}^{[2]}$ (5.15).

Hence, in $\widehat{a}^{[4]}$ (4.10) functions \overline{a}_0 , \overline{a}_1 , \widetilde{a}_1 , \widetilde{a}_2 , \widetilde{a}_3 , and \widetilde{a}_4 are given by (5.14)-(5.17) and (5.10)-(5.13), while \overline{a}_2 , \overline{a}_3 and \overline{a}_4 are to be found from further approximations. The substitution of $\widehat{a}^{[4]}$ into (5.4) produces the RHS-residual of order ε^5

$$\mathfrak{D}_3 \widehat{a}^{[4]} = \text{Res}[4] = O(\varepsilon^5) = O(1/\omega^{5/3}) \quad (5.19)$$

The residual of order $O(1/\omega^{5/3})$ for the operator \mathfrak{D}_3 means that the residual for original operators (2.1), (2.4), (2.34), and (5.3) is $O(1/\omega^{2/3})$.

5.2. Super-critical asymptotic procedure with $\alpha = 1/4$

Next we consider the oscillating velocity fields with first two terms of the drift velocity (4.26) vanishing

$$\overline{\mathbf{V}}_0 = \frac{1}{2} \langle [\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\xi}}] \rangle \equiv 0 \quad \text{and} \quad \overline{\mathbf{V}}_1 = \frac{1}{3} \langle [[\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\xi}}], \widetilde{\boldsymbol{\xi}}] \rangle \equiv 0 \quad (5.20)$$

Here one can find that the drift of magnitude $O(1)$ appears within the asymptotic family with $\alpha = 1/4$. Let us accept (5.20) and introduce a family of super-critical oscillations

$$\widehat{\mathbf{u}}(\mathbf{x}, t, \tau) = \widetilde{\mathbf{u}}(\mathbf{x}, t, \tau), \quad \delta = \omega^{-1/4} \quad (5.21)$$

where $\widetilde{\mathbf{u}}(\mathbf{x}, t, \tau) \in \mathbb{T} \cap \mathbb{O}(1)$ is a given function. Eqns. (2.34), (5.21) give:

$$\mathfrak{D} \widehat{a} = \omega \widehat{a}_\tau + \widehat{a}_t + \omega^{3/4} (\widetilde{\mathbf{u}} \cdot \nabla) \widehat{a} = 0 \quad (5.22)$$

The small parameter $\varepsilon_{1/4} = \omega^{-1/4}$ allows us to rewrite it as

$$\mathfrak{D}_4 \widehat{a} \equiv \widehat{a}_\tau + \varepsilon (\widetilde{\mathbf{u}} \cdot \nabla) \widehat{a} + \varepsilon^4 \widehat{a}_t = 0, \quad \mathfrak{D}_4 \equiv \varepsilon^4 \mathfrak{D} = \mathfrak{D}/\omega \quad (5.23)$$

where the subscript in $\varepsilon_{1/4}$ is dropped. We are looking for a solution of (5.23) in the form of regular series (4.4) with the redefined ε . The substitution of (4.4) into (5.23) produces the equations for successive approximations (*cf.* with (4.5)-(4.7) and (5.5)-(5.8))

$$\widehat{a}_{0\tau} = 0 \quad (5.24)$$

$$\widehat{a}_{1\tau} = -(\widetilde{\mathbf{u}} \cdot \nabla) \widehat{a}_0 \quad (5.25)$$

$$\widehat{a}_{2\tau} = -(\widetilde{\mathbf{u}} \cdot \nabla) \widehat{a}_1 \quad (5.26)$$

$$\widehat{a}_{3\tau} = -(\widetilde{\mathbf{u}} \cdot \nabla) \widehat{a}_2 \quad (5.27)$$

$$\widehat{a}_{n\tau} = -(\widetilde{\mathbf{u}} \cdot \nabla) \widehat{a}_{n-1} - \partial_t \widehat{a}_{n-4}, \quad n = 4, 5, \dots \quad (5.28)$$

Again, these equations can be solved in the most general form (see *Appendix B*). Here we present the truncated solution $\widehat{a}^{[4]}$ (4.10). For the tilde-part of each approximation

one can obtain explicit recurrent formulae

$$\tilde{a}_0 \equiv 0, \quad (5.29)$$

$$\tilde{a}_1 = -(\tilde{\xi} \cdot \nabla) \bar{a}_0, \quad (5.30)$$

$$\tilde{a}_2 = -(\tilde{\xi} \cdot \nabla) \bar{a}_1 - \{(\tilde{u} \cdot \nabla) \tilde{a}_1\}^\tau \quad (5.31)$$

$$\tilde{a}_3 = -(\tilde{\xi} \cdot \nabla) \bar{a}_2 - \{(\tilde{u} \cdot \nabla) \tilde{a}_2\}^\tau \quad (5.32)$$

$$\tilde{a}_4 = -(\tilde{\xi} \cdot \nabla) \bar{a}_3 - \{(\tilde{u} \cdot \nabla) \tilde{a}_3\}^\tau, \quad \forall \bar{a}_3 \quad (5.33)$$

and the transport equation for the averaged motion

$$(\partial_t + \bar{\mathbf{V}} \cdot \nabla) \bar{a} = 0, \quad (5.34)$$

$$\bar{\mathbf{V}} = \bar{\mathbf{V}}^{[2]} = \bar{\mathbf{V}}_2, \quad \bar{a} = \bar{a}^{[2]} = \bar{a}_0;$$

with the same notations as in (5.17), (5.18). Equation (5.34) means that averaged motion represents a pure drift with velocity $\bar{\mathbf{V}}_2$.

Hence, $\hat{a}^{[4]}$ can be written as (4.10) where $\bar{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3$, and \tilde{a}_4 are given by (5.29)-(5.34), while $\bar{a}_1, \bar{a}_2, \bar{a}_3$ and \bar{a}_4 are to be found from further approximations. The substitution of $\hat{a}^{[4]}$ into (5.23) produces the RHS-residual of order ε^5

$$\mathfrak{D}_4 \hat{a}^{[4]} = \text{Res}[4] = O(\varepsilon^5) = O(1/\omega^{5/4}) \quad (5.35)$$

The residual of order $O(1/\omega^{5/4})$ for the operator \mathfrak{D}_4 means that the residual for original operators (2.1), (2.4), (2.34), and (5.22) is $O(1/\omega^{1/4})$.

Remark: One can see that the results for both the super-critical families are simpler than for the critical one (*Sect.4*) and they are build up from the same elements: if the conditions of degenerations (5.1) or (5.20) are valid, then either $\bar{\mathbf{V}}_1$ or $\bar{\mathbf{V}}_2$ (4.26) becomes the leading $O(1)$ -term in (5.16) or (5.35). At the same time pseudo-diffusion does not appear (within the considered precision).

6. Polynomial Velocity for Critical and Super-Critical Oscillations.

Expressions (4.1), (5.2), (5.21) for velocity have a special form. One can expect that the adding of higher-order terms to these expressions may potentially change both the drift and pseudo-diffusion.

6.1. Critical asymptotic procedure with $\alpha = 1/2$

Let us consider a critical asymptotic family with a velocity in (2.2),(2.10),(2.34) given as (*cf.* with (4.1))

$$\hat{\mathbf{u}} = \hat{\mathbf{v}} + \frac{1}{\sqrt{\omega}} \hat{\mathbf{w}} + \frac{1}{\omega} \hat{\mathbf{r}}, \quad \delta = \omega^{-1/2}, \quad (6.1)$$

where $\hat{\mathbf{v}}, \hat{\mathbf{w}}, \hat{\mathbf{r}} \in \mathbb{H} \cap \mathbb{O}(1)$ are given functions. This formula might be seen as the first three terms of an infinite series, truncated by the omitting of terms $O(1/\omega^{3/2})$. Eqns. (2.34), (6.1) yield:

$$\mathfrak{D} \hat{a} = \omega \hat{a}_\tau + \sqrt{\omega} (\hat{\mathbf{v}} \cdot \nabla) \hat{a} + (\partial_t + \hat{\mathbf{w}} \cdot \nabla) \hat{a} + \frac{1}{\sqrt{\omega}} (\hat{\mathbf{r}} \cdot \nabla) \hat{a} = 0 \quad (6.2)$$

The small parameter $\varepsilon_{1/2} = 1/\sqrt{\omega}$ allows to rewrite it as

$$\mathfrak{D}_2 \hat{a} \equiv \hat{a}_\tau + \varepsilon (\hat{\mathbf{v}} \cdot \nabla) \hat{a} + \varepsilon^2 (\partial_t + \hat{\mathbf{w}} \cdot \nabla) \hat{a} + \varepsilon^3 (\hat{\mathbf{r}} \cdot \nabla) \hat{a} = 0 \quad (6.3)$$

where $\mathfrak{D}_2 \equiv \varepsilon^2 \mathfrak{D} = \mathfrak{D}/\omega$, and the subscript in $\varepsilon_{1/2}$ is dropped. We are looking for the solution of (6.3) in the form of regular series (4.4). The substitution of (4.4) into (6.3) produces the equations for successive approximations

$$\hat{a}_{0\tau} = 0 \quad (6.4)$$

$$\hat{a}_{1\tau} = -(\hat{\mathbf{v}} \cdot \nabla) \hat{a}_0 \quad (6.5)$$

$$\hat{a}_{2\tau} = -(\hat{\mathbf{v}} \cdot \nabla) \hat{a}_1 - (\partial_t + \hat{\mathbf{w}} \cdot \nabla) \hat{a}_0 \quad (6.6)$$

$$\hat{a}_{3\tau} = -(\hat{\mathbf{v}} \cdot \nabla) \hat{a}_2 - (\partial_t + \hat{\mathbf{w}} \cdot \nabla) \hat{a}_1 - (\hat{\mathbf{r}} \cdot \nabla) \hat{a}_0 \quad (6.7)$$

The solving of (6.4)-(6.7) follows the same steps as in *Appendix A*. An important new element is: the bar-part of eqn. (6.5) is $\bar{\mathbf{v}} \cdot \nabla \bar{a}_0 = 0$. It requires $\bar{\mathbf{v}} \equiv 0$ as a solvability condition. Therefore the leading term in the prescribed velocity (6.1) must always be purely oscillatory.

Here we present only the truncated solution $\hat{a}^{[3]}$ (4.10). For the tilde-parts \tilde{a}_k with $k = 0, 1, 2, 3$ the equations (6.4)-(6.7) yield the recurrent expressions

$$\tilde{a}_0 \equiv 0, \quad (6.8)$$

$$\tilde{a}_1 = -(\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_0, \quad (6.9)$$

$$\tilde{a}_2 = -\{(\tilde{\mathbf{v}} \cdot \nabla) \tilde{a}_1\}^\tau - (\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_1 - (\tilde{\boldsymbol{\eta}} \cdot \nabla) \bar{a}_0 \quad (6.10)$$

$$\tilde{a}_3 = -\{(\tilde{\mathbf{v}} \cdot \nabla) \tilde{a}_2\}^\tau - \{(\tilde{\mathbf{w}} \cdot \nabla) \tilde{a}_1\}^\tau - \quad (6.11)$$

$$-(\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_2 - (\tilde{\boldsymbol{\eta}} \cdot \nabla) \bar{a}_1 - (\tilde{\boldsymbol{\zeta}} \cdot \nabla) \bar{a}_0 - (\partial_t + \bar{\mathbf{w}} \cdot \nabla) \tilde{a}_1^\tau$$

$$\tilde{\boldsymbol{\xi}} \equiv \tilde{\mathbf{v}}^\tau, \quad \tilde{\boldsymbol{\eta}} \equiv \tilde{\mathbf{w}}^\tau, \quad \tilde{\boldsymbol{\zeta}} \equiv \tilde{\mathbf{r}}^\tau, \quad (6.12)$$

and for the bar-parts \bar{a}_k we obtain the transport equation

$$(\partial_t + (\bar{\mathbf{w}} + \bar{\mathbf{V}}_0) \cdot \nabla) \bar{a}_0 = 0 \quad (6.13)$$

$$(\partial_t + (\bar{\mathbf{w}} + \bar{\mathbf{V}}_0) \cdot \nabla) \bar{a}_1 + ((\bar{\mathbf{r}} + \bar{\mathbf{V}}_1 + \bar{\mathbf{V}}_{12}) \cdot \nabla) \bar{a}_0 = 0 \quad (6.14)$$

$$\bar{\mathbf{V}}_0 \equiv \frac{1}{2} \langle [\tilde{\mathbf{v}}, \tilde{\boldsymbol{\xi}}] \rangle, \quad \bar{\mathbf{V}}_1 \equiv \frac{1}{3} \langle [[\tilde{\mathbf{v}}, \tilde{\boldsymbol{\xi}}], \tilde{\boldsymbol{\xi}}] \rangle, \quad \bar{\mathbf{V}}_{12} \equiv \langle [\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}}] \rangle, \quad (6.15)$$

which can be rewritten (with the error $O(\varepsilon^2)$) as a single advection equation

$$(\partial_t + \bar{\mathbf{V}} \cdot \nabla) \bar{a} = 0 \quad (6.16)$$

$$\bar{\mathbf{V}} = (\bar{\mathbf{w}} + \bar{\mathbf{V}}_0) + \varepsilon(\bar{\mathbf{r}} + \bar{\mathbf{V}}_1 + \bar{\mathbf{V}}_{12}), \quad \bar{a} = \bar{a}_0 + \varepsilon \bar{a}_1 \quad (6.17)$$

Notice that the solution given by (6.8)-(6.17) satisfies eqn. (6.3) with $\text{Res}[3] = O(\omega^{-2})$.

Remark: In (6.4)-(6.17) we present only four steps ($k = 0, 1, 2, 3$) of successive approximations since that is enough to give us an instructive correction $\bar{\mathbf{V}}_{12}$ to the previously calculated drift (4.18), (4.20). The fifth step ($k = 4$) has been also carried out but for brevity it is not shown here: it produces a combination of a drift term $\bar{\mathbf{V}}_2$ (which is more cumbersome than (4.21)) and pseudo-diffusion *with the same coefficients* (4.23). The related calculations are similar to those of *Appendix A*.

6.2. Super-critical asymptotic procedure with $\alpha = 1/3$

Next, we consider a super-critical asymptotic family with a velocity similar to (6.1)

$$\hat{\mathbf{u}} = \hat{\mathbf{v}} + \omega^{-1/3} \hat{\mathbf{w}} + \omega^{-2/3} \hat{\mathbf{r}}, \quad \delta = \omega^{-1/3} \quad (6.18)$$

where $\hat{\mathbf{v}}, \hat{\mathbf{w}}, \hat{\mathbf{r}} \in \mathbb{H} \cap \mathbb{O}(1)$ are given functions. Expression (6.18) might be seen as the first three terms in an infinite series, truncated by omitting the terms $O(1/\omega)$. Eqns. (2.34),

(6.18) yield:

$$\mathfrak{D}\hat{a} = \omega\hat{a}_\tau + \omega^{2/3}(\hat{\mathbf{v}} \cdot \nabla)\hat{a} + \omega^{1/3}(\hat{\mathbf{w}} \cdot \nabla)\hat{a} + (\partial_t + \hat{\mathbf{r}} \cdot \nabla)\hat{a} = 0 \quad (6.19)$$

The small parameter $\varepsilon_{1/3} = \omega^{-1/3}$ (2.9) allows to rewrite (6.19) as

$$\mathfrak{D}_3\hat{a} \equiv \hat{a}_\tau + \varepsilon(\hat{\mathbf{v}} \cdot \nabla)\hat{a} + \varepsilon^2(\hat{\mathbf{w}} \cdot \nabla)\hat{a} + \varepsilon^3(\partial_t + \hat{\mathbf{r}} \cdot \nabla)\hat{a} = 0 \quad (6.20)$$

where $\mathfrak{D}_3 \equiv \varepsilon^3\mathfrak{D} = \mathfrak{D}/\omega$ and the subscript in $\varepsilon_{1/3}$ is dropped. We are looking for the solution of (6.20) in the form of regular series (4.4). The substitution of (4.4) into (6.20) produces the equations for successive approximations

$$\hat{a}_{0\tau} = 0 \quad (6.21)$$

$$\hat{a}_{1\tau} = -(\hat{\mathbf{v}} \cdot \nabla)\hat{a}_0 \quad (6.22)$$

$$\hat{a}_{2\tau} = -(\hat{\mathbf{v}} \cdot \nabla)\hat{a}_1 - (\hat{\mathbf{w}} \cdot \nabla)\hat{a}_0 \quad (6.23)$$

$$\hat{a}_{3\tau} = -(\hat{\mathbf{v}} \cdot \nabla)\hat{a}_2 - (\hat{\mathbf{w}} \cdot \nabla)\hat{a}_1 - (\partial_t + \hat{\mathbf{r}} \cdot \nabla)\hat{a}_0 \quad (6.24)$$

The solving of (6.21)-(6.24) follows the same steps as in *Appendix B*. There are two solvability conditions

$$\bar{\mathbf{v}} \equiv 0, \quad \bar{\mathbf{V}}_0 = -\bar{\mathbf{w}} \quad (6.25)$$

which are required for the existence of the bar-part solutions in (6.22) and (6.23). One might see the consideration of *Sect.5* as being physically incomplete. Indeed, one can expect that the results for small $\bar{\mathbf{V}}_0 \neq 0$ are close to that for $\bar{\mathbf{V}}_0 = 0$. This expectation has been met and fulfilled by a more general degeneration condition $\bar{\mathbf{V}}_0 = -\bar{\mathbf{w}}$ (6.25) which leads to a more general (than (5.15)) expression for a drift.

For tilde-parts \tilde{a}_k one can obtain the explicit recurrent expressions which coincide with (6.9)-(6.11), while the formula (6.12) for \tilde{a}_3 should be replaced by

$$\begin{aligned} \tilde{a}_3 = & -\{(\tilde{\mathbf{v}} \cdot \nabla)\tilde{a}_2\}^\tau - \{(\tilde{\mathbf{w}} \cdot \nabla)\tilde{a}_1\}^\tau - \\ & -(\tilde{\boldsymbol{\xi}} \cdot \nabla)\bar{a}_2 - (\tilde{\boldsymbol{\eta}} \cdot \nabla)\bar{a}_1 - (\tilde{\boldsymbol{\zeta}} \cdot \nabla)\bar{a}_0 - (\bar{\mathbf{w}} \cdot \nabla)\tilde{a}_1^\tau \end{aligned} \quad (6.26)$$

For the bar-parts one can derive the equation

$$(\partial_t + \bar{\mathbf{V}} \cdot \nabla)\bar{a}_0 = 0, \quad \bar{\mathbf{V}} = \bar{\mathbf{r}} + \bar{\mathbf{V}}_1 + \bar{\mathbf{V}}_{12} \quad (6.27)$$

which is valid with the error $O(\varepsilon^2)$ and contains the same $\bar{\mathbf{V}}_1$ and $\hat{\mathbf{V}}_{12}$ as in (6.15). Notice that this solution satisfies the equation (6.20) with $\text{Res}[3] = O(\omega^{-4/3})$.

Remarks:

1. One might additionally consider a polynomial version of velocity for $\omega^{1/4}$ -family (5.21). The related results could be predicted by the analogy between the material of *Sect.6.2* and *Sect.5*.

2. For the critical family (6.1) slow motion represents the drift (6.17) of magnitude $O(1)$ combined with the same pseudo-diffusion of order $O(1/\omega)$ as in (4.28), hence the results are very similar to those of *Sect.4*.

3. All the results of *Sects.2-6* can be generalized to an arbitrary flow domain Ω with fixed boundary $\partial\Omega$. Then (2.26) yields that

$$\bar{\mathbf{V}}_0 \cdot \mathbf{n} = \bar{\mathbf{V}}_1 \cdot \mathbf{n} = \bar{\mathbf{V}}_2 \cdot \mathbf{n} = \tilde{\mathbf{u}}' \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \quad (6.28)$$

provided $\hat{\mathbf{u}}$ satisfies the no-leak condition. One can also see that if a fluid is incompressible then (2.25) yields that all drift velocities in (4.17)-(4.24) are also incompressible:

$$\text{div} \tilde{\mathbf{u}} = 0 \quad \Rightarrow \quad \text{div} \bar{\mathbf{V}}_0 = \text{div} \bar{\mathbf{V}}_1 = \text{div} \bar{\mathbf{V}}_2 = 0 \quad (6.29)$$

The explicit expressions (4.17)-(4.22) show that due to incompressibility only the last

term in $\overline{\mathbf{V}}_2$ (4.21) vanishes. Similar (for (6.13)-(6.15)) the incompressibility yields

$$\operatorname{div} \tilde{\mathbf{v}} = \operatorname{div} \hat{\mathbf{w}} = \operatorname{div} \hat{\mathbf{r}} = 0 \quad \Rightarrow \quad \operatorname{div} \overline{\mathbf{V}}_0 = \operatorname{div} \overline{\mathbf{V}}_1 = \operatorname{div} \overline{\mathbf{V}}_{12} = 0 \quad (6.30)$$

4. For the super-critical family (6.18) with the degeneration (6.25) slow motion is a pure drift with the velocity (6.27) which is ‘upgraded’ to $O(1)$ from $O(\varepsilon)$ -term in (6.17). The qualitatively new addition to a drift (in comparison with a critical family) is a cross-term $\overline{\mathbf{V}}_{12} \equiv \langle [\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}}] \rangle$. It represents an averaged commutator between two mutually independent functions $\tilde{\mathbf{v}}$ and $\tilde{\boldsymbol{\eta}}$; hence $\overline{\mathbf{V}}_{12}$ expands the class of available analytical expressions for a drift. For example, for an incompressible fluid $\langle [\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}}] \rangle = \nabla \times \langle \tilde{\mathbf{v}} \times \tilde{\boldsymbol{\eta}} \rangle$, so one can see that $\langle \tilde{\mathbf{v}} \times \tilde{\boldsymbol{\eta}} \rangle$ represents an arbitrary vector-potential for $\overline{\mathbf{V}}_{12}$.

7. Examples.

7.1. Superposition of two modulated oscillatory fields of the same frequency

The velocity field $\tilde{\mathbf{u}}$ (4.1) and $\tilde{\boldsymbol{\xi}} \equiv \tilde{\mathbf{u}}^\tau$ (2.21) are

$$\tilde{\mathbf{u}}(\mathbf{x}, t, \tau) = \overline{\mathbf{p}}(\mathbf{x}, t) \sin \tau + \overline{\mathbf{q}}(\mathbf{x}, t) \cos \tau \quad (7.1)$$

$$\tilde{\boldsymbol{\xi}}(\mathbf{x}, t, \tau) = -\overline{\mathbf{p}}(\mathbf{x}, t) \cos \tau + \overline{\mathbf{q}}(\mathbf{x}, t) \sin \tau \quad (7.2)$$

with arbitrary \mathbb{B} -functions $\overline{\mathbf{p}}$ and $\overline{\mathbf{q}}$. The straightforward calculations yield

$$[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}] = [\overline{\mathbf{p}}, \overline{\mathbf{q}}] \quad (7.3)$$

so the commutator surprisingly is not oscillating. The drift velocities (4.20), (4.21) are

$$\overline{\mathbf{V}}_0 = \frac{1}{2} \langle [\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}] \rangle = \frac{1}{2} [\overline{\mathbf{p}}, \overline{\mathbf{q}}], \quad \overline{\mathbf{V}}_1 = \frac{1}{3} \langle [[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}], \tilde{\boldsymbol{\xi}}] \rangle \equiv 0, \quad (7.4)$$

$$\overline{\mathbf{V}}_2 = \frac{1}{8} ([\overline{\mathbf{P}}, \overline{\mathbf{p}}] + [\overline{\mathbf{Q}}, \overline{\mathbf{q}}]) - \frac{1}{4} ([\overline{\mathbf{p}}_t, \overline{\mathbf{p}}] + [\overline{\mathbf{q}}_t, \overline{\mathbf{q}}]) + \quad (7.5)$$

$$+ \frac{1}{4} (\overline{\mathbf{p}} \operatorname{div} \overline{\mathbf{P}}' + \overline{\mathbf{q}} \operatorname{div} \overline{\mathbf{Q}}' + \overline{\mathbf{P}}' \operatorname{div} \overline{\mathbf{p}} + \overline{\mathbf{Q}}' \operatorname{div} \overline{\mathbf{q}}),$$

$$\overline{\mathbf{P}} \equiv [\overline{\mathbf{V}}_0, \overline{\mathbf{p}}], \quad \overline{\mathbf{Q}} \equiv [\overline{\mathbf{V}}_0, \overline{\mathbf{q}}], \quad \overline{\mathbf{P}}' \equiv \overline{\mathbf{p}}_t - \overline{\mathbf{P}}, \quad \overline{\mathbf{Q}}' \equiv \overline{\mathbf{q}}_t - \overline{\mathbf{Q}},$$

$$\langle \tilde{\xi}_i \tilde{\xi}_k \rangle = \frac{1}{2} (\overline{p}_i \overline{p}_k + \overline{q}_i \overline{q}_k) \quad (7.6)$$

A pseudo-diffusion matrix $\overline{\kappa}_{ik}$, which follows after the substitution of (7.6) into (4.23).

Remarks:

1. The degeneration $\overline{\mathbf{V}}_0 \equiv 0$ in an incompressible fluid (6.29), (6.30) corresponds to $\overline{\mathbf{p}} \times \overline{\mathbf{q}} = \nabla \overline{\varphi}$ with an arbitrary potential $\overline{\varphi}(\mathbf{x}, t)$. For any such fields all results of *Sect. 5.1* are valid.

2. The velocity $\tilde{\mathbf{u}}$ (7.1) is general enough to produce any given function $\overline{\mathbf{V}}_0(\mathbf{x}, t)$. To obtain $\overline{\mathbf{p}}(\mathbf{x}, t)$ and $\overline{\mathbf{q}}(\mathbf{x}, t)$ one has to solve a bilinear first-order PDE

$$[\overline{\mathbf{p}}, \overline{\mathbf{q}}] = (\overline{\mathbf{q}} \cdot \nabla) \overline{\mathbf{p}} - (\overline{\mathbf{p}} \cdot \nabla) \overline{\mathbf{q}} = 2\overline{\mathbf{V}}_0(\mathbf{x}, t) \quad (7.7)$$

which represents an underdetermined bi-linear PDE-problem for two unknown functions.

7.2. Fourier series of modulated fields

The previous example can be generalized by the consideration of a velocity

$$\tilde{\mathbf{u}}(\mathbf{x}, t, \tau) = \sum_{k=1}^{\infty} \bar{\mathbf{p}}_k(\mathbf{x}, t) \sin k\tau + \bar{\mathbf{q}}_k(\mathbf{x}, t) \cos k\tau \quad (7.8)$$

$$\tilde{\boldsymbol{\xi}} = \sum_{k=1}^{\infty} -\frac{\bar{\mathbf{p}}_k}{k} \cos k\tau + \frac{\bar{\mathbf{q}}_k}{k} \sin k\tau \quad (7.9)$$

The calculations of $\bar{\mathbf{V}}_0$ lead to an infinite sum of commutators

$$\bar{\mathbf{V}}_0 = \frac{1}{2} \langle [\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}] \rangle = \sum_{k=1}^{\infty} \frac{1}{2k} [\bar{\mathbf{p}}_k, \bar{\mathbf{q}}_k] \quad (7.10)$$

In particular, (7.10) gives an infinite number of fields $\tilde{\mathbf{u}}$ with $\bar{\mathbf{V}}_0 \equiv 0$. For example, if $\bar{\mathbf{p}}_k \equiv 0, \forall k$ then

$$\bar{\mathbf{V}}_1 = \frac{1}{3} \langle [[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}], \tilde{\boldsymbol{\xi}}] \rangle = \frac{1}{12} \sum_{m \pm n \pm l = 0} \frac{1}{ml} [[\bar{\mathbf{q}}_n, \bar{\mathbf{q}}_m], \bar{\mathbf{q}}_l] \quad (7.11)$$

where the sum is taken over all sets of three positive integers m, n, l such that $m \pm n \pm l = 0$. It is clear that for many combinations $\bar{\mathbf{V}}_0 \equiv 0$ and $\bar{\mathbf{V}}_1 \neq 0$ (see *Sect.5.1*).

7.3. The Stokes drift

This most celebrated example of a drift we consider in some details. We start with the scaling (2.7), (2.8) and the limit (2.14). The dimensional solution for a plane potential travelling gravity wave (see Stokes (1847), Lamb (1932), Debnath (1994)) is

$$\hat{\mathbf{u}}^* = U \tilde{\mathbf{u}}, \quad U = \frac{k^* g^* h^*}{\omega^*}, \quad \tilde{\mathbf{u}} = \exp(k^* y^*) \begin{pmatrix} \cos(k^* x^* - \tau) \\ \sin(k^* x^* - \tau) \end{pmatrix} \quad (7.12)$$

where ω^*, k^*, h^* , and g^* are dimensional frequency, wavenumber, spatial amplitude, and gravity; (x^*, y^*) are Cartesian coordinates, and $\tau \equiv \omega^* t^*$. One can immediately notice that: (i) the characteristic length is $L = 1/k^*$; and (ii) the scale U is apparent. Hence the dimensionless scaling parameters (2.8), (2.12) and the asymptotic limit (2.14) appear as $k = 1$ and

$$\varepsilon_1 = 1/T\omega^* = 1/\omega \rightarrow 0, \quad \delta \equiv U/\omega^* L = g^* h^* / L^2 \omega^{*2} = gh/\omega^2 \rightarrow 0 \text{ as } \omega \rightarrow \infty \quad (7.13)$$

where one can choose the scale $T = T(\omega^*)$ in an arbitrary way, its only mission is to provide $\varepsilon_1 \rightarrow 0$ as $\omega \rightarrow \infty$ (see *Sect.2.3*). The simplest way to provide $\delta = 1/\sqrt{\omega}$ (4.1) is to choose an asymptotic family as

$$T = \text{const}, \quad L = \text{const}, \quad gh = O(\omega^{3/2}) \quad (7.14)$$

The dimensionless velocity field (7.12) and $\tilde{\boldsymbol{\xi}}$ are

$$\tilde{\mathbf{u}} = A e^{ky} \begin{pmatrix} \cos(kx - \tau) \\ \sin(kx - \tau) \end{pmatrix}, \quad \tilde{\boldsymbol{\xi}} = A e^{ky} \begin{pmatrix} -\sin(kx - \tau) \\ \cos(kx - \tau) \end{pmatrix} \quad (7.15)$$

where in the chosen system of units $A = 1$ and $k = 1$; however, we keep both A and k in the formulae for tracking its physical meaning. The fields $\bar{\mathbf{p}}(x, y), \bar{\mathbf{q}}(x, y)$ (7.1) are

$$\bar{\mathbf{p}} = A e^{ky} \begin{pmatrix} \sin kx \\ -\cos kx \end{pmatrix}, \quad \bar{\mathbf{q}} = A e^{ky} \begin{pmatrix} \cos kx \\ \sin kx \end{pmatrix} \quad (7.16)$$

The calculations (with the use of (7.3), (7.4)) yield

$$\overline{\mathbf{V}}_0 = kA^2 e^{2ky} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \overline{\mathbf{V}}_1 \equiv 0 \quad (7.17)$$

which gives $\overline{\mathbf{V}}_0$ proportional to the classical Stokes drift and a zero value for the first correction to it; the explicit formula for $\overline{\mathbf{V}}_2$ is not given here for brevity. At the same time

$$\langle \tilde{\xi}_i \tilde{\xi}_k \rangle = \Xi(x, y, t) \delta_{ik}, \quad \text{with} \quad \Xi = \frac{1}{2} A^2 e^{2ky} \quad (7.18)$$

which can be seen as ‘locally isotropic’ oscillations. The matrix of pseudo-diffusion (4.23) is

$$2\overline{\chi}_{ik} = \{ (\partial_t + \overline{\mathbf{V}}_0 \cdot \nabla) \delta_{ik} - 2\overline{\chi} \overline{e}_{ik} \} \Xi, \quad 2\overline{e}_{ik} \equiv \frac{\partial \overline{V}_{0i}}{\partial x_k} + \frac{\partial \overline{V}_{0k}}{\partial x_i}$$

Further calculations show that

$$\overline{\chi}_{ik} = -\overline{\chi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{with} \quad \overline{\chi} \equiv \frac{1}{4} k^2 A^4 e^{3ky}$$

One can see that the eigenvalues $\overline{\chi}_1 = -\overline{\chi}$ and $\overline{\chi}_2 = \overline{\chi}$ correspond to ordinary diffusion in one direction and anti-diffusion in the perpendicular direction, so here we have a mixed case of pseudo-diffusion. The averaged equation (4.25) (with an error $O(\varepsilon^3)$) can be written as

$$\begin{aligned} \overline{a}_t + (\overline{V}_0 + \varepsilon^2 \overline{V}_2) \overline{a}_x &= \varepsilon^2 (\overline{\chi}_y \overline{a}_x + \overline{\chi} \overline{a}_{xy}) \\ \overline{a} &= \overline{a}_0 + \varepsilon \overline{a}_1 + \varepsilon^2 \overline{a}_2 \end{aligned} \quad (7.19)$$

where \overline{V}_0 and \overline{V}_2 are the x -components of corresponding velocities (their y -components vanish). This equation has an exact solution $\overline{a} = \overline{a}(y)$ (with an arbitrary function $\overline{a}(y)$), which is not blurred by pseudo-diffusion.

Remarks:

1. In order to avoid confusion one should recall that the choice of asymptotic family (7.14) has nothing to do with physical relations between parameters. Any asymptotic family represents only a formal one-dimensional parametrization in the functional space of all possible functions $\hat{\mathbf{u}}^*$ (7.12). The only aim of such a parametrization is to build a valid asymptotic procedure. Therefore any particular wave (7.12) with given values $k^* = k_0^*$ and $\omega^* = \omega_0^*$ can be considered as a point on an asymptotic curve with $k = 1$ and with variable ω irrespectively to the presence of any dispersion relation between k_0^* and ω_0^* .

2. A parameter $gh \rightarrow \infty$ as $\omega \rightarrow \infty$ (7.14) which is a common property for many asymptotic procedures.

3. More general (than (7.14)) asymptotic families are considered in *Sect.2.3* and in (8.65), (8.66).

4. Both fields (7.15) are unbounded as $y \rightarrow \infty$, but it is not essential for our purposes.

7.4. Spherical ‘acoustic’ wave

A velocity potential for an outgoing spherical wave is

$$\tilde{\phi} = \frac{A}{r} \sin(kr - \tau) \quad (7.20)$$

where A , k , and r are an amplitude, a wavenumber, and a radius in a spherical coordinate system. The velocity is purely radial has a form (7.1)

$$\tilde{u} = \bar{p} \sin \tau + \bar{q} \cos \tau, \quad (7.21)$$

$$\bar{p} = A \left(\frac{1}{r^2} \cos kr + \frac{k}{r} \sin kr \right), \quad \bar{q} = A \left(-\frac{1}{r^2} \sin kr + \frac{k}{r} \cos kr \right) \quad (7.22)$$

where \tilde{u} , \bar{p} , and \bar{q} are radial components of corresponding vector-fields. The fields $\tilde{\xi}$ and $[\tilde{u}, \tilde{\xi}]$ are also purely radial; the radial component for the commutator is

$$\xi \tilde{u}_r - \tilde{u} \xi_r = A^2 k^3 / r^2 \quad (7.23)$$

where ξ is radial component of $\tilde{\xi}$ and subscript r stands for radial derivative. The drift (7.3),(7.4) is purely radial with

$$\bar{V}_0 = \frac{A^2 k^3}{2r^2}, \quad \bar{V}_1 = 0, \quad \bar{V}_2 = \frac{A^4 k^5}{16r^4} \left(3k^2 - \frac{5}{r^2} \right) \quad (7.24)$$

It is interesting that \bar{V}_0 formally coincides with the velocity caused by a point source in an incompressible fluid and for small r the convergence of \bar{V} is endangered, since \bar{V}_2 dominates over \bar{V}_0 . Further calculations yield

$$\langle \xi^2 \rangle = \frac{A^2}{2r^2} (k^2 + 1/r^2), \quad \bar{\chi} = A^4 k^5 / 4r^2 > 0 \quad (7.25)$$

where $\bar{\chi}$ stands for the only nonzero rr -component of $\bar{\chi}_{ik}$. One can see that in this case pseudo-diffusion appears as ordinary diffusion.

7.5. The \bar{V}_1 -drift.

If $\bar{V}_0 \equiv 0$ then the drift of order $O(1)$ is given by (5.15). Let the velocity field (5.2) be a superposition of two standing waves of frequencies ω and 2ω :

$$\tilde{u}(\mathbf{x}, t, \tau) = \bar{p}(\mathbf{x}, t) \sin \tau + \bar{q}(\mathbf{x}, t) \sin 2\tau \quad (7.26)$$

$$\tilde{\xi}(\mathbf{x}, t, \tau) = -\bar{p}(\mathbf{x}, t) \cos \tau - \frac{1}{2} \bar{q}(\mathbf{x}, t) \cos 2\tau \quad (7.27)$$

$$[\tilde{u}, \tilde{\xi}] = \frac{1}{2} [\bar{p}, \bar{q}] (2 \cos \tau \sin 2\tau - \cos 2\tau \sin \tau) \quad (7.28)$$

Hence (4.20) yields

$$\bar{V}_0 = \frac{1}{2} \langle [\tilde{u}, \tilde{\xi}] \rangle \equiv 0, \quad \bar{V}_1 = \frac{1}{3} \langle [[\tilde{u}, \tilde{\xi}], \tilde{\xi}] \rangle = \frac{1}{8} [[\bar{p}, \bar{q}], \bar{p}] \quad (7.29)$$

These expressions produce infinitely many examples of flows with $O(1)$ -drift in a supercritical asymptotic family (5.15).

7.6. A plane travelling wave of a general shape

The velocity field (7.1) for a plane wave in three-dimensional space is

$$\tilde{\mathbf{u}} = \mathbf{A} \tilde{f}'(\mathbf{k} \cdot \mathbf{x} - \tau) \quad (7.30)$$

where \mathbf{A} and \mathbf{k} are two constant vectors, \tilde{f} is a \mathbb{T} -function of a scalar variable, its τ -periodicity automatically leads to periodicity w.r.t. $\mathbf{k} \cdot \mathbf{x}$, primes stand for ordinary derivatives. Calculations yield

$$\bar{V}_0 = \mathbf{A}(\mathbf{A} \cdot \mathbf{k}) \langle \tilde{f}'^2 \rangle, \quad \bar{V}_1 = \mathbf{A}(\mathbf{A} \cdot \mathbf{k})^2 \langle \tilde{f}'^3 \rangle \quad (7.31)$$

It shows that the only case of a zero drift corresponds to a transversal wave $\mathbf{A} \perp \mathbf{k}$ (an incompressible fluid) and the maximal drift takes place for a longitudinal wave $\mathbf{A} \parallel \mathbf{k}$. A pseudo-diffusion matrix is

$$\bar{\chi}_{ik} \equiv 0 \quad \text{for} \quad \langle \tilde{\xi}_i \tilde{\xi}_k \rangle = A_i A_k \langle \tilde{f}^2 \rangle, \quad (7.32)$$

Next step is to consider the superposition of two travelling plane waves (7.30):

$$\begin{aligned} \tilde{\mathbf{u}} &= \mathbf{A} \tilde{f}'(\mathbf{k} \cdot \mathbf{x} - \tau) + \mathbf{B} \tilde{g}'(\mathbf{l} \cdot \mathbf{x} - \tau) = -\mathbf{A} \tilde{f}_\tau(\mathbf{k} \cdot \mathbf{x} - \tau) - \mathbf{B} \tilde{g}_\tau(\mathbf{l} \cdot \mathbf{x} - \tau) \\ \tilde{\boldsymbol{\xi}} &= -\mathbf{A} \tilde{f}_\tau(\mathbf{k} \cdot \mathbf{x} - \tau) - \mathbf{B} \tilde{g}_\tau(\mathbf{l} \cdot \mathbf{x} - \tau) \end{aligned} \quad (7.33)$$

with constant vectors \mathbf{A} , \mathbf{B} , \mathbf{k} , \mathbf{l} . Calculations yield

$$\bar{\mathbf{V}}_0 = (\mathbf{A} \cdot \mathbf{k}) \mathbf{A} \langle \tilde{f}'^2 \rangle + (\mathbf{B} \cdot \mathbf{k}) \mathbf{B} \langle \tilde{g}'^2 \rangle + ((\mathbf{A} \cdot \mathbf{l}) \mathbf{B} + (\mathbf{B} \cdot \mathbf{k}) \mathbf{A}) \langle \tilde{f}' \tilde{g}' \rangle \quad (7.34)$$

which exhibits the interference (third) term. If $f' = \sin(\mathbf{k} \cdot \mathbf{x} - \tau)$ and $g' = \sin(\mathbf{l} \cdot \mathbf{x} - \tau)$ then (7.34) gives

$$2\bar{\mathbf{V}}_0 = (\mathbf{A} \cdot \mathbf{k}) \mathbf{A} + (\mathbf{B} \cdot \mathbf{k}) \mathbf{B} + ((\mathbf{A} \cdot \mathbf{l}) \mathbf{B} + (\mathbf{B} \cdot \mathbf{k}) \mathbf{A}) \cos(\mathbf{k} - \mathbf{l}) \mathbf{x} \quad (7.35)$$

while $\bar{\mathbf{V}}_1$ is too cumbersome to be presented here. Also

$$2\langle \xi_i \xi_k \rangle = A_i A_k + B_i B_k + (A_i B_k + A_k B_i) \cos(\mathbf{k} - \mathbf{l}) \mathbf{x}$$

For two mutually perpendicular longitudinal waves $(\mathbf{A} \parallel \mathbf{k}) \perp (\mathbf{B} \parallel \mathbf{l})$ one can obtain

$$\bar{\chi}_{ik} = -\frac{1}{2} (A_i B_k + A_k B_i) [(\mathbf{A} \cdot \mathbf{k})^2 - (\mathbf{B} \cdot \mathbf{l})^2] \sin(\mathbf{k} - \mathbf{l}) \mathbf{x} \quad (7.36)$$

which for the plane flow $\mathbf{A} = (A, 0)$, $\mathbf{B} = (0, B)$ gives

$$\bar{\chi}_{ik} = -\frac{1}{8} AB [(\mathbf{A} \cdot \mathbf{k})^2 - (\mathbf{B} \cdot \mathbf{l})^2] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin(\mathbf{k} - \mathbf{l}) \mathbf{x} \quad (7.37)$$

Here we have the eigenvalues $\bar{\chi}_1 = -\bar{\chi}_2$ of opposite signs, they are also oscillating in space. Once again (as in *Sect. 7.3*), we deal with the sign-indefinite case of pseudo-diffusion. This example can be linked to plane sound waves; the drift (7.31) can give an addition to an acoustic streaming (see *e.g.* Lighthill (1978b)).

7.7. The drift for the polynomial velocity (6.18)

The expression (6.27) for the $O(1)$ -drift within a super-critical family contains the term given in (6.15)

$$\bar{\mathbf{V}}_{12} \equiv \langle [\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}}] \rangle \quad (7.38)$$

which is built by two mutually independent oscillating functions. Such a functional freedom allows to construct broad functional classes of drifts. However, in this case one should keep in mind that the degeneration condition $\bar{\mathbf{V}}_0 = -\bar{\boldsymbol{\omega}}$ (6.25) imposes a restriction on the coefficients of (6.18). For example, one can chose $\tilde{\mathbf{v}}$ such that $\bar{\mathbf{V}}_0 = -\bar{\boldsymbol{\omega}} = 0$. In this case the function $\tilde{\mathbf{v}}$ in (7.38) is not arbitrary. Alternatively, one can consider $\bar{\mathbf{V}}_0 = -\bar{\boldsymbol{\omega}}$ as the definition of $\bar{\boldsymbol{\omega}}$. In this case the functions $\tilde{\mathbf{v}}$ and $\tilde{\boldsymbol{\eta}}$ in (7.38) are indeed mutually independent.

7.8. The Bjorknes configuration of two pulsating point sources

This example is aimed to clarify the global structure of drift motion in an oscillating flow related to Cook (1882), Hicks (1879), Hicks (1890) and to show that interesting

oscillating flow, which are different from waves, do exist. An incompressible velocity from the class of flows (7.1) in Cartesian coordinates \mathbf{x} is given by

$$\bar{\mathbf{q}}(\mathbf{x}) = \nabla \frac{1}{|\mathbf{x}|}, \quad \bar{\mathbf{p}}(\mathbf{x}) = \nabla \frac{1}{|\mathbf{y}|}, \quad \mathbf{y} = \mathbf{x} - \mathbf{l}, \quad \mathbf{l} = \text{const} \quad (7.39)$$

which represents a superposition of two oscillating point sources. Calculations yield a rotationally symmetric w.r.t. the \mathbf{l} -axis field:

$$\bar{\mathbf{V}}_0 = \frac{2(\mathbf{x} \cdot \mathbf{y})}{(|\mathbf{x}|^2 |\mathbf{y}|^2)^2} (|\mathbf{y}|^2 \mathbf{x} - |\mathbf{x}|^2 \mathbf{y}), \quad \bar{\mathbf{V}}_0 \cdot (\mathbf{x} \times \mathbf{y}) = 0 \quad (7.40)$$

Let us introduce cylindrical coordinates (ρ, ϕ, z) with an origin at $\mathbf{x} = \mathbf{l}/2$. The related ODEs (8.3) are

$$\begin{aligned} \dot{\rho} &= M\rho z, \quad \dot{\phi} = 0, \quad \dot{z} = -\frac{M}{2} \left(\frac{l^2}{4} - z^2 + \rho^2 \right) \\ l &\equiv |\mathbf{l}|, \quad M \equiv -36 \frac{\rho^2 + z^2 - l^2/4}{|\mathbf{x}|^2 |\mathbf{y}|^2} \\ |\mathbf{x}|^2 &= \rho^2 + (z - l/2)^2, \quad |\mathbf{y}|^2 = \rho^2 + (z + l/2)^2 \end{aligned} \quad (7.41)$$

The qualitative dynamics of particles described by this system is: (i) two singularities at the points $\mathbf{x} = \pm \mathbf{l}/2$; (ii) any point of the sphere $\rho^2 + z^2 = l^2/4$ represents an equilibrium; and (iii) the directions of particle motions inside and outside of this sphere are topologically opposite to each other.

7.9. The complexity of particle dynamics for $\bar{\mathbf{V}}_0$

Now we illustrate the complexity of particle dynamics (8.3) for the zero-order drift velocity $\bar{\mathbf{V}}_0$ (4.20) (see Aref (1984), Ottino (1989), Samelson & Wiggins (2006)). Let an incompressible velocity (7.1) be

$$\bar{\mathbf{p}} = \begin{pmatrix} \cos y \\ 0 \\ \sin y \end{pmatrix}, \quad \bar{\mathbf{q}} = \begin{pmatrix} a \sin z \\ b \sin x + a \cos z \\ b \cos x \end{pmatrix} \quad (7.42)$$

where (x, y, z) are Cartesian coordinates, a, b are constants. Either of this fields, taken separately, produces simple integrable dynamics of particles. Calculations yield

$$\bar{\mathbf{V}}_0 = \begin{pmatrix} -a \sin y \sin x - 2b \sin y \cos z \\ b \sin z \sin y - a \cos x \cos y \\ b \cos z \cos y + 2a \sin x \cos y \end{pmatrix}, \quad (7.43)$$

Straightforward computations for this steady flow exhibit chaotic dynamics of particles. In particular, positive Lyapunov exponents have been observed. This example shows, that the drift created by simple oscillatory field can produce complex dynamics. Since the averaged dynamics is chaotic then related results by Arnold (1964), Aref (1984), Ottino (1989), Samelson & Wiggins (2006), Chierchia & Gallavotti (1994) can be applied to it. This example brings up new questions: (i) what is the relationship between chaotic motions for the original dynamical system and the averaged one? (ii) can the oscillatory part of a solution also cause chaotic dynamics? (iii) can a chaotic drift and the pseudo-diffusion of *Sect.4* complement each other? (iv) how a chaotic drift can be used in the theory of mixing? This example has been constructed and computed by A.B.Morgulis (private communications) for the use in this paper.

7.10. The appearance of pseudo-diffusion (PD) due to slow-time-modulations

Let us consider a simple example that clarifies the meaning of pseudo-diffusion.

An infinite fluid oscillates as a rigid body with the small displacement of any material particle $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(t, \tau)$. The Eulerian coordinate of a particle is

$$\mathbf{x} = \bar{\mathbf{x}} + \tilde{\mathbf{x}}, \quad \langle \tilde{\mathbf{x}} \rangle \equiv 0, \quad \tilde{\mathbf{x}} = \varepsilon \tilde{\mathbf{x}}_1, \quad \tilde{\mathbf{x}}_1 \in \mathbb{O}(1) \quad (7.44)$$

where $\langle \cdot \rangle$ is the τ -average (2.17); the small parameter ε and the function $\tilde{\mathbf{x}}_1(t, \tau)$ can be either chosen by us or taken from (8.14)-(8.16). A drift for a rigid-body oscillations is absent $\bar{\mathbf{V}} \equiv 0$, hence the mean coordinate $\bar{\mathbf{x}}$ of a particle is not changing with time. For simplicity we also accept that the mean coordinate $\bar{\mathbf{x}}$ coincides with the initial (Lagrangian) coordinate \mathbf{X} of a particle, hence $\tilde{\mathbf{x}}(0, 0) = 0$.

A distribution of a Lagrangian tracer \hat{a} is given as $\hat{a} = A(\mathbf{X})$. Then

$$\hat{a}(\mathbf{x}, t, \tau) = A(\mathbf{x} - \varepsilon \tilde{\mathbf{x}}_1) \quad (7.45)$$

One can expand both sides of (7.45) as

$$\hat{a}_0 + \varepsilon \hat{a}_1 + \varepsilon^2 \hat{a}_2 + \dots = A(\mathbf{x}) - \varepsilon \tilde{x}_{1i} \frac{\partial A(\mathbf{x})}{\partial x_i} + \frac{\varepsilon^2}{2} \tilde{x}_{1i} \tilde{x}_{1k} \frac{\partial^2 A(\mathbf{x})}{\partial x_i \partial x_k} + \dots \quad (7.46)$$

where $\hat{a}_n = \hat{a}_n(\mathbf{x}, t, \tau) = \bar{a}_n(\mathbf{x}, t) + \tilde{a}_n(\mathbf{x}, t, \tau)$. The average $\langle \cdot \rangle^{\mathbf{x}}$ of this equation yields

$$\bar{a}_0 + \varepsilon \bar{a}_1 + \varepsilon^2 \bar{a}_2 + \dots = A(\mathbf{x}) + \frac{\varepsilon^2}{2} \langle \tilde{\xi}_i \tilde{\xi}_k \rangle \frac{\partial^2 A(\mathbf{x})}{\partial x_i \partial x_k} + \dots \quad (7.47)$$

where we have changed $\tilde{\mathbf{x}}_1(t, \tau)$ to $\tilde{\boldsymbol{\xi}}(t, \tau)$ (4.16), which is valid for the given precision. The t -differentiation of (7.47) gives

$$\bar{a}_{0t} + \varepsilon \bar{a}_{1t} + \varepsilon^2 \bar{a}_{2t} + \dots = \frac{\varepsilon^2}{2} \langle \tilde{\xi}_i \tilde{\xi}_k \rangle_t \frac{\partial^2 a_0(\mathbf{x})}{\partial x_i \partial x_k} + \dots \quad (7.48)$$

where we have used $A = \bar{a}_0$ (which follows from (7.47)). One can see that eqns. (4.17)-(4.19) taken for zero drift $\bar{\mathbf{V}} = \bar{\mathbf{V}}_0 + \varepsilon \bar{\mathbf{V}}_1 + \varepsilon^2 \bar{\mathbf{V}}_2 + \dots \equiv 0$ coincide with (7.48).

It means that in this particular case *PD* represents a diffusion-like term that appears in the averaged equations as a correction \bar{a}_2 caused by the curvature of the function $\bar{a}_0 = \bar{a}_0(\mathbf{x})$. In order to make this explanation clear one can imagine a one-dimensional case of (7.44) with $A = A(Z)$ in (7.45) for a single Lagrangian coordinate Z . Then the averaging of oscillations of the entire graph $A = A(Z)$ in Z -direction evidently produces a second-order correction (proportional to the curvature A_{ZZ}) to the distribution $\bar{a}_0(z) = A(z)$ which occur in the absence of oscillations. Now one might assume that for general oscillating flows the nature of pseudo-diffusion is the same:

Conjecture: Pseudo-diffusion in (4.19), (4.25) is always caused by the oscillations of Lagrangian tracer $\hat{a}(\mathbf{X}, t, \tau)$ with respect to fixed Eulerian coordinates \mathbf{x} .

According to this conjecture the presence of a drift produces only a logically natural change from $\partial/\partial t$ (for flows (7.45), (7.48) without a drift) to the ‘material’ derivatives (with the drift velocity $\bar{\mathbf{V}}$ in (4.17)-(4.19)). Hence this conjecture looks reliable, it indicates that pseudo-diffusion represents a natural as well as necessary term in the averaged equations.

Remark: The effects of pseudo-diffusion and diffusion (or anti-diffusion, or anisotropic diffusion-anti-diffusion) are described by the same equations, hence they are mathematically equivalent to each other. However, pseudo-diffusion is used only with regular asymptotic procedures.

7.11. *The sub-critical family of oscillations: procedure with $\alpha = 1$*

Let us consider an example of a sub-critical asymptotic family with $\alpha = 1$ and $\beta = 0$. For this family $U = L/T = \text{const}$ in (2.13) which might be seen as an advantage for some applications. Following (3.10) we accept that (2.33) is:

$$\hat{\mathbf{u}}(\mathbf{x}, t, \tau) = \tilde{\mathbf{u}}(\mathbf{x}, t, \tau), \quad \delta = \omega^{-1}, \quad (7.49)$$

Then (2.34) yields:

$$\mathfrak{D}\hat{a} = \omega\hat{a}_\tau + \hat{a}_t + (\tilde{\mathbf{u}} \cdot \nabla)\hat{a} = 0 \quad (7.50)$$

The small parameter $\varepsilon_1 = 1/\omega$ (2.9) allows to rewrite it as

$$\mathfrak{D}_1\hat{a} \equiv \hat{a}_\tau + \varepsilon(\hat{a}_t + \tilde{\mathbf{u}} \cdot \nabla)\hat{a} = 0, \quad \mathfrak{D}_1 \equiv \varepsilon\mathfrak{D} = \mathfrak{D}/\omega \quad (7.51)$$

where the subscript in ε_1 has been dropped. We are looking for the solution of (7.51) in the form of regular series (4.4) with redefined ε . The substitution of (4.4) into (7.51) produces the equations of successive approximations

$$\hat{a}_{0\tau} = 0 \quad (7.52)$$

$$\hat{a}_{n\tau} = -(\partial_t + \tilde{\mathbf{u}} \cdot \nabla)\hat{a}_{n-1}, \quad \partial_t \equiv \partial/\partial t, \quad n = 1, 2, 3, \dots \quad (7.53)$$

The solving of (7.52),(7.53) yields

$$\tilde{a}_0 \equiv 0, \quad (7.54)$$

$$\tilde{a}_1 = -(\tilde{\xi} \cdot \nabla)\bar{a}_0, \quad (7.55)$$

$$\tilde{a}_2 = -(\tilde{\xi} \cdot \nabla)\bar{a}_1 - \{(\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_1\}^\tau - \tilde{a}_{1t}^\tau, \quad (7.56)$$

$$\bar{a}_{0t} = 0, \quad (7.57)$$

$$\bar{a}_{1t} + (\bar{\mathbf{V}}_0 \cdot \nabla)\bar{a}_0 = 0 \quad (7.58)$$

$$\bar{a}_{2t} + (\bar{\mathbf{V}}_0 \cdot \nabla)\bar{a}_1 + (\bar{\mathbf{V}}_1^+ \cdot \nabla)\bar{a}_0 = \frac{\partial}{\partial x_i} \left(\bar{\chi}_{ik}^{\text{sub}} \frac{\partial \bar{a}_0}{\partial x_k} \right), \quad (7.59)$$

$$\bar{\mathbf{V}}_1^{\text{sub}} = \bar{\mathbf{V}}_1 + \frac{1}{2}\langle [\tilde{\xi}, \tilde{\xi}_t] \rangle + \frac{1}{2}\langle \tilde{\xi} \text{div} \tilde{\xi} \rangle_t, \quad 2\bar{\chi}_{ik}^{\text{sub}} \equiv \langle \tilde{\xi}_i \tilde{\xi}_k \rangle_t, \quad (7.60)$$

where $\bar{\mathbf{V}}_0$ and $\bar{\mathbf{V}}_1$ are the same as in (4.20). Equations (7.57)-(7.59) can be written as a single advection-pseudo-diffusion equation (valid with the error $O(\varepsilon^3)$)

$$(\partial_t + \bar{\mathbf{V}} \cdot \nabla)\bar{a} = \frac{\partial}{\partial x_i} \left(\bar{\kappa}_{ik}^{\text{sub}} \frac{\partial \bar{a}}{\partial x_k} \right) \quad (7.61)$$

$$\bar{\mathbf{V}} = \varepsilon\bar{\mathbf{V}}_0 + \varepsilon^2\bar{\mathbf{V}}_1^{\text{sub}}, \quad \bar{\kappa}_{ik}^{\text{sub}} = \varepsilon^2\bar{\chi}_{ik}^{\text{sub}} \quad (7.62)$$

$$\bar{a} = \bar{a}_0 + \varepsilon\bar{a}_1 + \varepsilon^2\bar{a}_2 \quad (7.63)$$

Eqn. (7.61) shows that the averaged motion of \hat{a} represents a *pseudo-drift* with velocity $\bar{\mathbf{V}}$ and *pseudo-diffusion* with the matrix-coefficient $\bar{\kappa}_{ik}$. The reason for introducing the term *pseudo-drift* is the same as we used for *pseudo-diffusion* in (4.25): the drift-like terms in (7.58),(7.59),(7.61) play a part of sources, known from previous approximations.

Remarks to *Sect. 7.11*:

1. A simplified (in comparison with (4.23)) expression for pseudo-diffusivity in (7.59) complies with the interpretation of pseudo-diffusivity of previous subsection.

2. The system (7.57)-(7.60) produces mainly diverging solutions. Indeed, (7.57) yields $\bar{a}_0 = \bar{a}_0(\mathbf{x})$. Let $\bar{a}_0(\mathbf{x}) \neq \text{const}$, then (7.58) gives a linear with t growth of \bar{a}_1 for any t -independent $\bar{\mathbf{V}}_0$, for example for the Stokes drift (see *Sect. 7.3*). If we choose $\bar{a}_0 \equiv \text{const}$, then a similar growth appears for \bar{a}_2 etc. A statement that any sub-critical asymptotic

procedure with $\bar{a}_t \in \mathbb{O}(1)$ produces diverging solutions might be considered as a conjecture, but its analysis is beyond the scope of this paper.

3. This equations of this subsection do not produce the transport equation (drift) in any approximation.

4. The arguments of items 2 and 3 emphasize the significance of critical and super-critical asymptotic procedures (of *Sects.2-7*).

Remarks to all *Sect.7*:

1. From the examples of *Sects.7.1-7.9* one can see that the expressions for the drift velocity $\bar{\mathbf{V}}$ (4.20), (4.21), (6.17) can provide arbitrary functional form and magnitude not higher than $O(1)$. In order to make further progress one should prescribe some particular oscillating flows (2.2).

2. These examples also show that pseudo-diffusivity in (4.25),(4.26) can appear in three qualitatively different cases: (i) all positive eigenvalues $\bar{\chi}_i$ corresponds to ordinary diffusion; (ii) all negative $\bar{\chi}_i$ corresponds to anti-diffusion, and (iii) the mixed signs correspond to more complex anisotropic evolution with diffusion in some directions and anti-diffusion in the others. The complexity of the problem (4.25) increases if one considers the dependence of $\bar{\chi}_i$ on \mathbf{x} and t . The general theory of related linear PDEs (*e.g.* (7.19)) has not been developed yet, see Polyanin (2002), Myint-U & Debnath (1987).

3. The number of our examples is naturally restricted, therefore *Sect.7* illustrates our studies of *Sects.3-6* only partially.

8. Links to Other Theories

8.1. *TTAM-solution of characteristic equation*

We are not aware about any paper which calculates several approximations for a drift by solving the related ODE by two-timing method. Therefore we present the *TTAM*-versions of such calculations here. Let us return to the original dimensional equation (2.1), (2.2) with an oscillating velocity

$$\hat{\mathbf{u}}^* = \hat{\mathbf{u}}^*(\hat{\mathbf{x}}^*, t^*, \tau) \quad (8.1)$$

where $\hat{\mathbf{x}}^*$ is an upgraded notation for original Eulerian coordinates which is introduced here instead of \mathbf{x}^* (see *Sect.2*) since Eulerian coordinates in Lagrangian description also represent the hat-functions (2.15). The oscillating trajectories

$$\hat{\mathbf{x}}^* = \hat{\mathbf{x}}^*(\mathbf{X}^*, t^*, \tau) \quad (8.2)$$

(which represent the characteristics for the hyperbolic equation (2.1), (2.4)) can be found by solving Cauchy's problem for an ODE

$$\frac{d\hat{\mathbf{x}}^*}{ds^*} = \hat{\mathbf{u}}^*(\hat{\mathbf{x}}^*, t^*, \tau), \quad \hat{\mathbf{x}}^*|_{t=0} = \mathbf{X}^* \quad (8.3)$$

$$t^* = s^*, \quad \tau = \omega^* s^*; \quad \frac{d}{ds^*} = \frac{\partial}{\partial t^*} + \omega^* \frac{\partial}{\partial \tau} \quad (8.4)$$

where \mathbf{X}^* is the Lagrangian coordinate of a fluid particle and $t^* = 0$ automatically leads to $\tau = 0$. Using the same procedure as for (2.10) we can obtain the dimensionless form of (8.3)

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}(t, \tau) : \quad \frac{d\hat{\mathbf{x}}}{ds} = \omega \delta \hat{\mathbf{u}}(\hat{\mathbf{x}}, t, \tau); \quad t = s, \quad \tau = \omega s, \quad \frac{d}{ds} = \frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \tau} \quad (8.5)$$

which represents the two-timing form of a dynamical system for the motion of particles, see Arnold (1964), Aref (1984), Ottino (1989), Samelson & Wiggins (2006). We accept

(2.33) and introduce a test-solution for our inspection procedure

$$\hat{\mathbf{x}}(t, \tau) = \bar{\mathbf{x}}_0(t) + \frac{1}{\omega^\alpha} \hat{\mathbf{x}}_1(t, \tau), \quad \alpha > 0; \quad \delta = \omega^{\beta-1}, \quad \beta < 1 \quad (8.6)$$

that is motivated similarly to (3.1) (if one takes $\tilde{\mathbf{x}}_0 \neq 0$, then τ ceases to be a fast variable). The substitution of (8.6) into (8.5) yields

$$\left(\omega \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} \right) \left(\bar{\mathbf{x}}_0 + \frac{1}{\omega^\alpha} \hat{\mathbf{x}}_1 \right) = \omega^\beta \tilde{\mathbf{u}} \left(\bar{\mathbf{x}}_0 + \frac{1}{\omega^\alpha} \hat{\mathbf{x}}_1 \right) \quad (8.7)$$

The decomposing of the RHS into Taylor's series with the retaining of two leading terms gives

$$\bar{\mathbf{x}}_{0t} + \omega^{1-\alpha} \tilde{\mathbf{x}}_{1\tau} + \omega^{-\alpha} \hat{\mathbf{x}}_{1t} = \omega^\beta \tilde{\mathbf{u}} + \omega^{\beta-\alpha} (\hat{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}} \quad (8.8)$$

where $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\bar{\mathbf{x}}(t, \tau), t, \tau)$. The bar- and tilde- parts of this equation are

$$\bar{\mathbf{x}}_{0t} + \omega^{-\alpha} \bar{\mathbf{x}}_{1t} = \omega^{\beta-\alpha} \langle (\tilde{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}} \rangle \quad (8.9)$$

$$\omega^{1-\alpha} \tilde{\mathbf{x}}_{1\tau} + \omega^{-\alpha} \hat{\mathbf{x}}_{1t} = \omega^\beta \tilde{\mathbf{u}} + \omega^{\beta-\alpha} (\bar{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}} + \omega^{\beta-\alpha} \{ (\tilde{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}} \} \quad (8.10)$$

where we have used the brace notation (2.28). The leading terms in these equations are

$$\bar{\mathbf{x}}_{0t} = \omega^{\beta-\alpha} \langle (\tilde{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}} \rangle \quad (8.11)$$

$$\omega^{1-\alpha} \tilde{\mathbf{x}}_{1\tau} = \omega^\beta \tilde{\mathbf{u}} \quad (8.12)$$

Here the τ -average $\langle \cdot \rangle$ does not carry any superscript, since the spatial independent variable is absent; at the same time $\tilde{\mathbf{u}}$ depends on an unknown function $\bar{\mathbf{x}}_0(t)$. The equation (8.12) gives $\alpha + \beta = 1$, after that (8.11) brings us back to the same notions of super-critical, critical, and sub-critical asymptotic families (3.8)-(3.10). For a critical family $\alpha = \beta = 1/2$ and $\varepsilon = \varepsilon_{1/2} = 1/\sqrt{\omega}$, which produces an asymptotic problem

$$\left(\omega \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} \right) \hat{\mathbf{x}} = \sqrt{\omega} \tilde{\mathbf{u}}, \quad \hat{\mathbf{x}} = \bar{\mathbf{x}}_0 + \frac{1}{\sqrt{\omega}} \hat{\mathbf{x}}_1 + \frac{1}{\omega} \hat{\mathbf{x}}_2 + \dots \quad (8.13)$$

or

$$\tilde{\mathbf{x}}_\tau = -\varepsilon^2 \hat{\mathbf{x}}_t + \varepsilon \tilde{\mathbf{u}}(\hat{\mathbf{x}}, t, \tau), \quad \hat{\mathbf{x}} = \bar{\mathbf{x}}_0 + \varepsilon \hat{\mathbf{x}}_1 + \varepsilon^2 \hat{\mathbf{x}}_2 + \dots \quad (8.14)$$

The successive approximations are

$$\tilde{\mathbf{x}}_{0\tau} = 0 \quad (8.15)$$

$$\tilde{\mathbf{x}}_{1\tau} = \tilde{\mathbf{u}}_0 \quad (8.16)$$

$$\tilde{\mathbf{x}}_{2\tau} + \bar{\mathbf{x}}_{0t} = (\hat{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}}_0 \quad (8.17)$$

$$\tilde{\mathbf{x}}_{3\tau} + \tilde{\mathbf{x}}_{1t} = (\hat{\mathbf{x}}_2 \cdot \nabla) \tilde{\mathbf{u}}_0 + \frac{1}{2} \hat{x}_{1i} \hat{x}_{1k} \frac{\partial^2 \tilde{\mathbf{u}}_0}{\partial \bar{x}_{0i} \partial \bar{x}_{0k}} \quad (8.18)$$

$$\tilde{\mathbf{x}}_{4\tau} + \tilde{\mathbf{x}}_{2t} = (\tilde{\mathbf{x}}_3 \cdot \nabla) \tilde{\mathbf{u}}_0 + \tilde{x}_{1i} \tilde{x}_{2k} \frac{\partial^2 \tilde{\mathbf{u}}_0}{\partial \bar{x}_{0i} \partial \bar{x}_{0k}} + \frac{1}{6} \tilde{x}_{1i} \tilde{x}_{1k} \tilde{x}_{1j} \frac{\partial^3 \tilde{\mathbf{u}}_0}{\partial \bar{x}_{0i} \partial \bar{x}_{0k} \partial \bar{x}_{0j}} \quad (8.19)$$

where $\tilde{\mathbf{u}}_0 \equiv \tilde{\mathbf{u}}(\bar{\mathbf{x}}_0, t, \tau)$. The system (8.15)-(8.19) can be solved using the same method as (4.5)-(4.7) and (8.56)-(8.59). Here we only present the solution for the averaged motion

$$\bar{\mathbf{x}}_{0t} = \bar{\mathbf{U}}_0 \quad (8.20)$$

$$\begin{aligned} \bar{\mathbf{x}}_{1t} &= (\bar{\mathbf{x}}_1 \cdot \nabla) \bar{\mathbf{U}}_0 + \bar{\mathbf{U}}_1 \\ \bar{\mathbf{x}}_{2t} &= (\bar{\mathbf{x}}_1 \cdot \nabla) \bar{\mathbf{U}}_1 + (\bar{\mathbf{x}}_2 \cdot \nabla) \bar{\mathbf{U}}_0 + \bar{\mathbf{U}}_2 \\ \bar{\mathbf{U}}_0 &= \langle (\tilde{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}} \rangle, \end{aligned} \quad (8.21)$$

$$\bar{\mathbf{U}}_1 = -\langle (\tilde{\mathbf{x}}_1 \cdot \nabla) (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{x}}_1 \rangle = \frac{1}{3} \langle [[\tilde{\mathbf{u}}, \tilde{\mathbf{x}}_1], \tilde{\mathbf{x}}_1] \rangle \quad (8.22)$$

$$\bar{\mathbf{U}}_2 = \langle (\tilde{\mathbf{x}}_3 \cdot \nabla) \tilde{\mathbf{u}} \rangle + \left\langle \tilde{x}_{1i} \tilde{x}_{2k} \frac{\partial^2 \tilde{\mathbf{u}}(\bar{\mathbf{x}})}{\partial \bar{x}_i \partial \bar{x}_k} \right\rangle - \frac{1}{2} \left\langle \tilde{x}_{1i} \tilde{x}_{1k} \tilde{u}_{1j} \frac{\partial^3 \tilde{\mathbf{x}}(\bar{\mathbf{x}})}{\partial \bar{x}_i \partial \bar{x}_k \partial \bar{x}_j} \right\rangle \quad (8.23)$$

which can be rewritten as

$$\begin{aligned} \bar{\mathbf{x}}_t &= \bar{\mathbf{U}} + (\bar{\mathbf{l}} \cdot \nabla) \bar{\mathbf{U}} \\ \bar{\mathbf{x}} &= \bar{\mathbf{x}}_0 + \varepsilon \bar{\mathbf{x}}_1 + \varepsilon^2 \bar{\mathbf{x}}_2 + \dots, \quad \bar{\mathbf{l}} \equiv \bar{\mathbf{x}} - \bar{\mathbf{x}}_0 \\ \bar{\mathbf{U}} &= \bar{\mathbf{U}}_0 + \varepsilon \bar{\mathbf{U}}_1 + \varepsilon^2 \bar{\mathbf{U}}_2 + \dots \end{aligned} \quad (8.24)$$

One can see that the system (8.24) describes two kinds of motion: in one every material point moves with velocity $\bar{\mathbf{U}}$; in the other every small material arc $\delta \bar{\mathbf{l}}$ is stretched by the same velocity field according to a standard law, see Batchelor (1967).

8.2. TTAM-approach to GLM-kinematics

The most general formula for a drift was obtained in *GLM*-theory by Andrews & McIntyre (1978), which is referred hereafter as *AM*. The direct comparison of the results of *Sects.2-7* with those of *AM* is not feasible due to the following conceptual differences: (i) *GLM*-theory in *AM* is not fully adapted to the two-timing method; (ii) *AM* uses one small parameter (the amplitude of $\tilde{\mathbf{x}}$) while we operate with two small parameters; (iii) *AM* employs a different averaging operation; and (iv) *AM* expresses a drift in terms of *generalized Lagrangian displacements* $\tilde{\mathbf{x}}$ (8.37), not a given velocity field (as in our case). We develop here the *TTAM*-version of *GLM*-kinematics with the aim to compare it with the results of *Sects.2-7*.

At the beginning of this subsection we use dimensional variables, but for brevity we suppress the asterisks; the use of dimensionless variables (starting from (8.48)) will be additionally notified. The core of the whole *GLM*-theory is a transformation of (8.3) into a PDE. In order to perform this transformation, we switch to mutually independent t, τ (see *Comment C* to (2.16)) and introduce a τ -averaged trajectory (8.2) as

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{X}, t) = \langle \hat{\mathbf{x}} \rangle^{\mathbf{X}} \quad (8.25)$$

where $\langle \cdot \rangle^{\mathbf{X}}$ emphasises that the τ -average (2.17) is performed for fixed \mathbf{X} . From (8.3) we get $\bar{\mathbf{x}}(\mathbf{X}, 0) = \mathbf{X}$. Equations (8.3), (8.25) give us three pairs of main kinematical items:

(i) two kinds of trajectories for a selected particle

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{X}, t, \tau) \text{ - the family of exact trajectories,} \quad (8.26)$$

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{X}, t) \text{ - the averaged trajectory,} \quad (8.27)$$

$$\text{with } \hat{\mathbf{x}}(\mathbf{X}, 0, 0) = \bar{\mathbf{x}}(\mathbf{X}, 0) = \mathbf{X}, \quad (8.28)$$

(ii) two kinds of velocities

$$\hat{\mathbf{u}} \equiv \partial \hat{\mathbf{x}} / \partial s|_{\mathbf{X}} \text{ - the family of exact velocities of a particle,} \quad (8.29)$$

$$\bar{\mathbf{v}} \equiv \partial \bar{\mathbf{x}} / \partial s|_{\mathbf{X}} \text{ - the averaged velocity of a particle}$$

(iii) two Eulerian expressions for the same material derivative

$$\partial/\partial s|_{\mathbf{X}} = \partial/\partial s|_{\hat{\mathbf{x}}} + \hat{\mathbf{u}} \cdot \hat{\nabla} = \partial/\partial s|_{\bar{\mathbf{x}}} + \bar{\mathbf{v}} \cdot \bar{\nabla} \quad (8.30)$$

Relations (8.26)-(8.30) show that for Lagrangian description (\mathbf{X}, t) one can use two different Eulerian descriptions $(\hat{\mathbf{x}}, s)$ and $(\bar{\mathbf{x}}, s)$ where $\bar{\mathbf{x}}$ and $\bar{\mathbf{v}}$ are called *the GLM-coordinate* and *GLM-velocity*. One also should keep in mind that in (8.29),(8.30) according to the chain rule

$$\partial/\partial s|_{\mathbf{X}} = \partial/\partial t|_{\mathbf{X}, \tau} + \omega \partial/\partial \tau|_{\mathbf{X}, t} \quad (8.31)$$

and similar equalities for fixed $\hat{\mathbf{x}}$ or $\bar{\mathbf{x}}$. The existence of one-to-one mapping between \mathbf{X} and $\hat{\mathbf{x}}$ represents a key postulate of classical fluid dynamics; it guarantees the existence of the unique inverse to (8.26) function

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{X}, t, \tau) \Leftrightarrow \mathbf{X} = \mathbf{X}(\hat{\mathbf{x}}, t, \tau); \quad 0 < J_0 < \infty, \quad J_0 \equiv \frac{\partial(\hat{x}_1, \hat{x}_2, \hat{x}_3)}{\partial(X_1, X_2, X_3)} \quad (8.32)$$

with a Jacobian J_0 . After that the function

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}(\hat{\mathbf{x}}, t, \tau) \quad (8.33)$$

can be obtained by the completely legal substitution of $\mathbf{X} = \mathbf{X}(\hat{\mathbf{x}}, t, \tau)$ (8.32) into $\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{X}, t)$ (8.25).

The additional assumption (specific for *GLM*) is an assumption of invertibility of (8.27)

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{X}, t) \Rightarrow \mathbf{X} = \mathbf{X}(\bar{\mathbf{x}}, t); \quad 0 < J_1 < \infty, \quad J_1 \equiv \frac{\partial(\bar{x}_1, \bar{x}_2, \bar{x}_3)}{\partial(X_1, X_2, X_3)} \quad (8.34)$$

The coordinate $\bar{\mathbf{x}}$ is defined by the averaging operation (8.25), hence it does not represent any physical motion; (8.34) represents an assumption which can be valid only under some additional restrictions. For the *GLM*-theory this invertibility is absolutely required; let us also accept it and show how it leads to *GLM*-kinematics.

The substitution of the inverse function $\mathbf{X} = \mathbf{X}(\bar{\mathbf{x}}, t)$ (8.34) into $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{X}, t, \tau)$ (8.26) produces the key (for the *GLM*-theory) relation between $\hat{\mathbf{x}}$ and $\bar{\mathbf{x}}$

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau); \quad 0 < J_2 < \infty, \quad J_2 \equiv \frac{\partial(\hat{x}_1, \hat{x}_2, \hat{x}_3)}{\partial(\bar{x}_1, \bar{x}_2, \bar{x}_3)} \quad (8.35)$$

that is inverse to (8.33). The functions (8.33), (8.35) give us one-to-one mapping between two systems of Eulerian coordinates $\hat{\mathbf{x}}$ and $\bar{\mathbf{x}}$; since $J_2 = J_0 J_1^{-1}$, the conditions for the Jacobians in (8.34) and (8.35) are equivalent to each other, provided (8.32) is valid.

An important property of the averaging operation (8.25)

$$\langle \cdot \rangle^{\bar{\mathbf{x}}} = \langle \cdot \rangle^{\mathbf{X}}, \quad (8.36)$$

follows from the fact that τ is not involved into transformations (8.34),(8.25) between \mathbf{X} into $\bar{\mathbf{x}}$. In other words, the τ -averaging operation commutes with the changing of variables (8.34). Taking $\langle \cdot \rangle^{\bar{\mathbf{x}}}$ of (8.35) we get

$$\hat{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau) = \bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau) \quad \text{with} \quad \langle \hat{\mathbf{x}} \rangle^{\bar{\mathbf{x}}} = \bar{\mathbf{x}}, \quad \langle \tilde{\mathbf{x}} \rangle^{\bar{\mathbf{x}}} = 0 \quad (8.37)$$

where $\langle \hat{\mathbf{x}} \rangle^{\bar{\mathbf{x}}} = \bar{\mathbf{x}}$ is valid by virtue of (8.25) and (8.36). Simultaneously (8.37) represents the definition of $\tilde{\mathbf{x}}$ which automatically possesses zero τ -average; in *AM* $\tilde{\mathbf{x}}$ is called *generalized Lagrangian displacement*.

Applying two Eulerian forms of material derivative (8.30) to both sides of (8.37) we obtain

$$(\partial/\partial s|_{\hat{\mathbf{x}}} + \hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{x}} = \hat{\mathbf{u}}(\hat{\mathbf{x}}, t, \tau) = (\partial/\partial s|_{\bar{\mathbf{x}}} + \bar{\mathbf{v}} \cdot \bar{\nabla})(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau)) \quad (8.38)$$

which can be rewritten as

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \tau} + \bar{\mathbf{v}} \cdot \bar{\nabla} \right) (\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau)) = \hat{\mathbf{u}}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) \quad \text{or} \quad (8.39)$$

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \tau} + \bar{\mathbf{v}} \cdot \bar{\nabla} \right) \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau) = \hat{\mathbf{u}}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) - \bar{\mathbf{v}}(\bar{\mathbf{x}}, t) \quad (8.40)$$

where (8.39) represents the equation for characteristics (8.3) written as a PDE in variables $\bar{\mathbf{x}}, t, \tau$. We call either (8.39) or (8.40) the two-timing form of *Andrews-McIntyre-Kinematics-Equation (AMKE)*; it contains unknown functions $\tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau)$ and $\bar{\mathbf{v}}(\bar{\mathbf{x}}, t)$. The description of fluid kinematics by (8.39) is rather unusual: *AMKE* describes an effective medium which moves with the averaged velocity $\bar{\mathbf{v}}$ that includes both an advective velocity and a drift. The exact motion of a material particle in this medium is described by two different fields: the averaged motion is described by *GLM*-velocity $\bar{\mathbf{v}}(\bar{\mathbf{x}}, t)$ and oscillatory displacement (from the *GLM*-position $\bar{\mathbf{x}}$) is given by $\tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau)$.

The first step in solving (8.40) is straightforward: its bar-part $\langle \cdot \rangle^{\bar{\mathbf{x}}}$ gives us *GLM*-velocity $\bar{\mathbf{v}}$ expressed in the terms of an original velocity

$$\bar{\mathbf{v}}(\bar{\mathbf{x}}, t) = \langle \hat{\mathbf{u}}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) \rangle^{\bar{\mathbf{x}}}, \quad (8.41)$$

which represents the main kinematical formula by *AM*. The average operation in the RHS of (8.41) is called *GLM-averaging*. *GLM*-averaging of a function $\hat{a}(\hat{\mathbf{x}}, t, \tau)$ is denoted by *AM* as

$$\bar{a}^L = \bar{a}^L(\mathbf{x}, t) \equiv \langle \hat{a}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) \rangle^{\bar{\mathbf{x}}} \quad (8.42)$$

The advection of a Lagrangian marker is described by the equation (2.1)

$$\left(\partial / \partial s|_{\hat{\mathbf{x}}} + \hat{\mathbf{u}}(\hat{\mathbf{x}}, t, \tau) \cdot \hat{\nabla} \right) \hat{a}(\hat{\mathbf{x}}, t, \tau) = 0 \quad (8.43)$$

Due to (8.30), we replace a material derivative in (8.43) with $\partial / \partial t|_{\bar{\mathbf{x}}} + \bar{\mathbf{v}} \cdot \bar{\nabla}$ and use (8.37)

$$(\partial / \partial s|_{\bar{\mathbf{x}}} + \bar{\mathbf{v}} \cdot \bar{\nabla}) \hat{a}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) = 0 \quad (8.44)$$

Then $\langle \cdot \rangle^{\bar{\mathbf{x}}}$ yields

$$(\partial / \partial t|_{\bar{\mathbf{x}}} + \bar{\mathbf{v}} \cdot \bar{\nabla}) \bar{a}^L = 0 \quad (8.45)$$

This remarkable equation says that the *GLM*-average \bar{a}^L (8.42) in the *GLM*-coordinates $\bar{\mathbf{x}}$ (8.25), (8.27) is purely advected with the *GLM*-velocity $\bar{\mathbf{v}}$ (8.41). *The drift* by *AM* is

$$\bar{\mathbf{U}}(\bar{\mathbf{x}}, t) = \bar{\mathbf{v}}(\bar{\mathbf{x}}, t) - \langle \hat{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) \rangle^{\bar{\mathbf{x}}} = \langle \hat{\mathbf{u}}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) \rangle^{\bar{\mathbf{x}}} - \langle \hat{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) \rangle^{\bar{\mathbf{x}}} \quad (8.46)$$

For a purely oscillating field $\hat{\mathbf{u}} = \tilde{\mathbf{u}}$ the average $\langle \hat{\mathbf{u}} \rangle^{\bar{\mathbf{x}}} \equiv 0$, hence

$$\bar{\mathbf{U}} = \bar{\mathbf{v}}(\bar{\mathbf{x}}, t) = \langle \hat{\mathbf{u}}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) \rangle^{\bar{\mathbf{x}}} \quad (8.47)$$

To follow the route used by *AM* we can take $\tilde{\mathbf{x}}$ of a small amplitude and decompose (8.46) and (8.47) into Taylor's series; however in the two-timing case this problem contains two small parameters and is richer in asymptotic procedures. In order to describe them we write the dimensionless form of *AMKE* (8.39) which is obtained by the same procedure as (2.10)

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \tau} + \bar{\mathbf{v}} \cdot \bar{\nabla} \right) (\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau)) = \omega \delta \tilde{\mathbf{u}}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) \quad (8.48)$$

where the given velocity field is taken as a purely oscillatory one. Starting from (8.48) and further in this section we use only dimensionless variables and functions, keeping the

same notations without asterisks. Using the same inspection procedure as in *Sect.3* we accept (2.33) and introduce a test-solution (similar to (3.1))

$$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \tilde{\mathbf{x}} = \bar{\mathbf{x}} + \frac{1}{\omega^\alpha} \tilde{\mathbf{y}}(\bar{\mathbf{x}}, t, \tau), \quad \alpha = \text{const} > 0 \quad (8.49)$$

where $\bar{\mathbf{x}} \in \mathbb{B} \cap \mathbb{O}(1)$, $\tilde{\mathbf{y}} \in \mathbb{T} \cap \mathbb{O}(1)$ and the amplitude of an oscillating part is given by the small parameter $\varepsilon_\alpha = 1/\omega^\alpha$ (2.9). Hence we obtain

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \tau} + \bar{\nu} \cdot \bar{\nabla} \right) \left(\bar{\mathbf{x}} + \frac{1}{\omega^\alpha} \tilde{\mathbf{y}}(\bar{\mathbf{x}}, t, \tau) \right) = \omega^\beta \tilde{\mathbf{u}} \left(\bar{\mathbf{x}} + \frac{1}{\omega^\alpha} \tilde{\mathbf{y}}(\bar{\mathbf{x}}, t, \tau) \right) \quad (8.50)$$

Decomposing the right-hand side of (8.50) into Taylor's series and omitting the terms of order $O(1/\omega^{2\alpha})$ and above, we have

$$\bar{\nu} + \omega^{1-\alpha} \tilde{\mathbf{y}}_\tau + \frac{1}{\omega^\alpha} D_{\bar{\nu}} \tilde{\mathbf{y}} = \omega^\beta \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) + \omega^{\beta-\alpha} (\tilde{\mathbf{y}} \cdot \nabla) \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) \quad (8.51)$$

The bar-part of this equation is

$$\bar{\nu} = \omega^{\beta-\alpha} \langle (\tilde{\mathbf{y}} \cdot \nabla) \tilde{\mathbf{u}} \rangle \quad (8.52)$$

while the difference between (8.51) and (8.52) produces the tilde-part of the equation. To make a meaningful equation out of the tilde-part (*cf.* with (3.4)) it is necessary to accept that dominating terms are of the same order:

$$\alpha + \beta = 1 \quad \text{and} \quad \tilde{\mathbf{y}}_\tau = \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) \quad (8.53)$$

Equations (8.52) and (8.53) brings us back to the same notions of super-critical, critical, and sub-critical asymptotic families (3.8)-(3.10) where a critical family is characterised by $\alpha = \beta$ (a classical *AM* case corresponds to a sub-critical family with $\alpha = 1$ and $\beta = 0$). Then *AMKE* (8.39) for the critical asymptotic family $\alpha = \beta = 1/2$ takes form

$$\begin{aligned} \left(\omega \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} + \bar{\mathbf{U}} \cdot \bar{\nabla} \right) (\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau)) &= \sqrt{\omega} \hat{\mathbf{u}}(\bar{\mathbf{x}} + \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau), t, \tau) \quad (8.54) \\ \tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau) &= \frac{1}{\sqrt{\omega}} \tilde{\mathbf{x}}_1(\bar{\mathbf{x}}, t, \tau) + \frac{1}{\omega} \tilde{\mathbf{x}}_2(\bar{\mathbf{x}}, t, \tau) + \dots \\ \bar{\mathbf{U}}(\bar{\mathbf{x}}, t) &= \bar{\mathbf{U}}_0(\bar{\mathbf{x}}, t) + \frac{1}{\sqrt{\omega}} \bar{\mathbf{U}}_1(\bar{\mathbf{x}}, t) + \frac{1}{\omega} \bar{\mathbf{U}}_2(\bar{\mathbf{x}}, t) + \dots \end{aligned}$$

The use of $\varepsilon = \varepsilon_{1/2} = 1/\sqrt{\omega}$ (2.9) transforms (8.54) into

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} + \varepsilon^2 \frac{\partial}{\partial t} + \varepsilon^2 (\bar{\mathbf{U}}_0 + \varepsilon \bar{\mathbf{U}}_1 + \varepsilon^2 \bar{\mathbf{U}}_2 + \dots) \cdot \bar{\nabla} \right) (\bar{\mathbf{x}} + \varepsilon \tilde{\mathbf{x}}_1 + \varepsilon^2 \tilde{\mathbf{x}}_2 + \dots) = \\ = \varepsilon \hat{\mathbf{u}}(\bar{\mathbf{x}} + \varepsilon \tilde{\mathbf{x}}_1 + \varepsilon^2 \tilde{\mathbf{x}}_2 + \dots, t, \tau) \end{aligned} \quad (8.55)$$

The first four approximations of this equation are:

$$\tilde{\mathbf{x}}_{1\tau} = \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) \quad (8.56)$$

$$\tilde{\mathbf{x}}_{2\tau} + \bar{\mathbf{U}}_0 = (\tilde{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) \quad (8.57)$$

$$\tilde{\mathbf{x}}_{3\tau} + \tilde{\mathbf{x}}_{1t} + \bar{\mathbf{U}}_1 + (\bar{\mathbf{U}}_0 \cdot \nabla) \tilde{\mathbf{x}}_1 = (\tilde{\mathbf{x}}_2 \cdot \nabla) \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) + \frac{1}{2} \tilde{x}_{1i} \tilde{x}_{1k} \frac{\partial^2 \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau)}{\partial \bar{x}_i \partial \bar{x}_k} \quad (8.58)$$

$$\begin{aligned} \tilde{\mathbf{x}}_{4\tau} + \tilde{\mathbf{x}}_{2t} + \bar{\mathbf{U}}_2 + (\bar{\mathbf{U}}_1 \cdot \nabla) \tilde{\mathbf{x}}_1 + (\bar{\mathbf{U}}_0 \cdot \nabla) \tilde{\mathbf{x}}_2 = \\ = (\tilde{\mathbf{x}}_3 \cdot \nabla) \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau) + \tilde{x}_{1i} \tilde{x}_{2k} \frac{\partial^2 \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau)}{\partial \bar{x}_i \partial \bar{x}_k} + \frac{1}{6} \tilde{x}_{1i} \tilde{x}_{1k} \tilde{x}_{1j} \frac{\partial^3 \tilde{\mathbf{u}}(\bar{\mathbf{x}}, t, \tau)}{\partial \bar{x}_i \partial \bar{x}_k \partial \bar{x}_j} \end{aligned} \quad (8.59)$$

These equations can be solved in the same way as the equations in *Sect.4*. The solutions are

$$\tilde{\mathbf{x}}_1 = \tilde{\mathbf{u}}^\tau \quad (8.60)$$

$$\tilde{\mathbf{x}}_2 = \{(\tilde{\mathbf{x}}_1 \cdot \nabla) \tilde{\mathbf{u}}\}^\tau \quad (8.61)$$

$$\tilde{\mathbf{x}}_3 = \tilde{\mathbf{x}}_{1t}^\tau + (\overline{\mathbf{U}}_0 \cdot \nabla) \tilde{\mathbf{x}}_1^\tau + \{(\tilde{\mathbf{x}}_2 \cdot \nabla) \tilde{\mathbf{u}}\}^\tau + \frac{1}{2} \left\{ \tilde{x}_{1i} \tilde{x}_{1k} \frac{\partial^2 \tilde{\mathbf{u}}(\overline{\mathbf{x}})}{\partial \overline{x}_i \partial \overline{x}_k} \right\}^\tau \quad (8.62)$$

with the same $\overline{\mathbf{U}}_0, \overline{\mathbf{U}}_1$, and $\overline{\mathbf{U}}_2$ as in (8.21)-(8.23).

Remarks and discussion:

1. In (8.48) we are looking for the dimensionless drift of order one $\overline{\mathbf{v}} = O(1)$ that (as we know from *Sects.4* and *5*) is true for critical and super-critical oscillations. The more systematic (but more cumbersome) approach is to replace $\overline{\mathbf{v}}$ in (8.48) by $\omega \delta \overline{\mathbf{v}}$ (which is required by (8.47)) and to derive rigorously that the first non-vanishing term in $\omega \delta \overline{\mathbf{v}}$ is $O(1)$.

2. The super-critical versions of the problems considered in *Sects.8.1,8.2* can be solved similarly to the problems in *Sects.5,6*.

3. The expression (8.20),(8.21) was obtained by Yudovich (2006).

4. The theories of both *Sect.8.1* and *Sect.8.2* operate with the average taken for a fixed Lagrangian coordinate \mathbf{X} . Indeed, it is the only possibility for the problem (8.5): this problem describes dynamics of a single particle. As to *AMKE* (8.39), it explicitly uses the average for fixed $\overline{\mathbf{x}}$ which is the same as for fixed \mathbf{X} , see (8.36).

5. One can observe a strong resemblance between the system (8.15)-(8.19) and the system (8.56)-(8.59). In (8.15)-(8.19) terms with $\overline{\mathbf{U}}_0, \overline{\mathbf{U}}_1, \dots$ in LHS are absent and Taylor's series in RHS contain $\hat{\mathbf{x}} = \overline{\mathbf{x}} + \tilde{\mathbf{x}}$ (instead of $\tilde{\mathbf{x}}$ in (8.56)-(8.59)). These differences are due to the fact that $\overline{\mathbf{x}}$ is an independent variable in (8.56)-(8.59) but in (8.15)-(8.19) it play a part of an unknown function. However one can show that these two systems are mathematically equivalent to each other, and the drift velocity $\overline{\mathbf{U}}$ in *Sect.8.1* and in *Sect.8.2* is the same. These properties are natural, since *AMKE* (8.39) represents a transformed ODE for characteristics (8.3) (see (8.25)-(8.39)).

6. Two sets of formulae for drift velocities (4.16), (4.20) and (8.60), (8.21),(8.22) look identical if one makes a correspondence

$$\begin{aligned} \tilde{\mathbf{u}}(\mathbf{x}, t, \tau) &\leftrightarrow \tilde{\mathbf{u}}(\overline{\mathbf{x}}, t, \tau), & \tilde{\boldsymbol{\xi}}(\mathbf{x}, t, \tau) &\leftrightarrow \tilde{\boldsymbol{\xi}}(\overline{\mathbf{x}}, t, \tau), \\ \overline{\mathbf{V}}_0(\mathbf{x}, t) &\leftrightarrow \overline{\mathbf{U}}_0(\overline{\mathbf{x}}, t), & \overline{\mathbf{V}}_1(\mathbf{x}, t) &\leftrightarrow \overline{\mathbf{U}}_1(\overline{\mathbf{x}}, t) \end{aligned} \quad (8.63)$$

hence one may conclude that our formulae for the zeroth and first approximations of a drift are the same as *AM*. However this resemblance is misleading, at least partially. The crucial point is the use of the *GLM*-coordinates $\overline{\mathbf{x}}$ (8.27) in *GLM*-kinematics *vs.* the original Eulerian coordinates $\mathbf{x} \equiv \hat{\mathbf{x}}$ (8.26) in *Sects.2-7*. It leads to the different definitions of the averaging operations with fixed $\overline{\mathbf{x}}$ *vs.* fixed \mathbf{x} . For example, the expressions $\tilde{\mathbf{x}}_1 = \tilde{\mathbf{u}}^\tau$ (8.60) and $\tilde{\boldsymbol{\xi}} \equiv \tilde{\mathbf{u}}^\tau$ (4.16) look identical, however they are different since the former is expressed as a function of $\overline{\mathbf{x}}$ and the latter as a function of $\mathbf{x} \equiv \hat{\mathbf{x}}$: these variables are constant during the \mathbb{T} -integrations. The implications of such a difference lay beyond the scope of this paper. At this stage one can see that: (i) the similarity between the expressions in (8.63) indeed corresponds to the close results for the zeroth and first approximations (due to the smallness of the difference $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \overline{\mathbf{x}}$); (ii) the expressions for $\overline{\mathbf{U}}_2$ and $\overline{\mathbf{V}}_2$ do not exhibit the similarity shown in (8.63).

7. One can see that, in order to express (8.45) in the spirit of *Sects.2-7*, \bar{a}^L has to be

decomposed into Taylor's series for small $\tilde{\mathbf{x}}$. The resulted equations is

$$(\partial/\partial t|_{\bar{\mathbf{x}}} + \bar{\mathbf{v}} \cdot \bar{\nabla}) \bar{R} = 0, \quad (8.64)$$

$$\bar{R} \equiv \langle \hat{a}(\bar{\mathbf{x}}, t, \tau) \rangle^{\bar{\mathbf{x}}} + \left\langle \tilde{x}_i \frac{\partial \hat{a}(\bar{\mathbf{x}}, t, \tau)}{\partial \bar{x}_i} \right\rangle^{\bar{\mathbf{x}}} + \frac{1}{2} \left\langle \tilde{x}_i \tilde{x}_k \frac{\partial^2 \hat{a}(\bar{\mathbf{x}}, t, \tau)}{\partial \bar{x}_i \partial \bar{x}_k} \right\rangle^{\bar{\mathbf{x}}} + \dots$$

where $\tilde{\mathbf{x}}(\bar{\mathbf{x}}, t, \tau)$, $\bar{\mathbf{v}}(\bar{\mathbf{x}}, t)$, and $\hat{a} = \bar{a} + \tilde{a}$ must be expressed as in (8.54) and (4.4). Then the terms of the type $\langle \tilde{x}_i \partial \tilde{a} / \partial \bar{x}_i \rangle$ should be calculated with the use of the full equation (8.44). Such calculations will produce Riemann's invariant \bar{R} which is constant along the averaged characteristic curves (or \bar{R} is transported with the drift velocity).

8. It is apparent that the establishing of one-to-one correspondence between (4.25) and (8.64) is not feasible. Indeed, (4.25) has a 'dissipative' pseudo-diffusion term which can not be incorporated in the transport equation (8.64). This fact should be expected from *Sect. 7.10*, where we have shown that pseudo-diffusion appears due to oscillations of Lagrangian coordinates with respect to fixed Eulerian coordinates. Hence, pseudo-diffusion (which is a product of Eulerian averaging) is not compatible with (8.64) (which is a product of Lagrangian averaging, see the *item 4* above). This incompatibility complies with a general understanding that different averaging operations preserve and loose original information differently. One can conclude that the averaged fields in *TTAM*-form of *GLM*-theory and in the theory of *Sects. 2-7* contain complementary (or just additional to each other) information. It opens the opportunity to look for the advantages given by the presence of these two descriptions.

8.3. Links to other classical theories

The classical drift velocity. The classical expression for a drift velocity is given by the formulae (26) of Longuet-Higgins (1953) or by (5.13.21) of Batchelor (1967). Our $\bar{\mathbf{V}}_0$ (4.20) will be proportional to this expression if it is rewritten with the use of τ -periodicity and integration by parts. In fact, the conjugated form $\langle (\tilde{\boldsymbol{\xi}} \cdot \nabla) \tilde{\mathbf{v}} \rangle$ of $\bar{\mathbf{V}}_0 = -\langle (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\boldsymbol{\xi}} \rangle$ (A 17) coincides with the classical expression. Consequently $\bar{\mathbf{V}}_0$ gives the correct expression for the drift in a potential travelling wave (Stokes (1847)). At the same time, there are four main differences between our results and that of the quoted classical papers: (i) we describe all involved variables and fields precisely, within the two-timing framework; (ii) we employ two small parameters instead of one; (iii) we obtain different magnitudes for a drift while in the classical paper the leading term is quadratic in small amplitude (which is also obtainable within our approach); (iv) we obtain two more formulae for drift velocities as well as find pseudo-diffusion.

The Krylov-Bogoliubov method. The Krylov-Bogoliubov averaging method (*KBAM*), see Bogoliubov & Mitropolskii (1961), Krylov & Bogoliubov (1947), Sanders & Verhulst (1985), is aimed to solve problems for special classes of ODEs with oscillating coefficients. This method can be straightforwardly used for the calculating of the averaged equations of characteristic curves (8.5). We have exploited this option: the required calculations are rather cumbersome, therefore we formulate here the results only. The first *KBAM*-term for a drift velocity coincides with $\bar{\mathbf{U}}_0$ (8.20), while the obtaining of the next two terms requires substantially more *KBAM* analytical calculations than *TTAM* ones. Here we make two comments: (1) *KBAM* as well as *TTAM* calculations of *Sect. 8.1* give the averaged equations for characteristic curves. Within these two methods applied to an ODE the problem of finding the averaged Riemann invariants cannot be addressed. (2) The areas of applicability of *TTAM* are much broader than *KBAM*: for example *TTAM* can operate with PDEs and incorporate molecular diffusivity to the governing equations (see *Sect. 9*).

The homogenization theory. TTAM has the same methodological roots as the method of homogenization (see Bensoussan, Lions and Papanicolaou (1978), Berdichevsky, Jikov, and Papanikolaou (1999)), which represents a version of the two-scale method. The long-wave ‘homogenization’ solutions for the transport of scalar admixture obtained by McLaughlin, Papanicolaou & Pironneau (1985), Vergassola & Avellaneda (1997), Frisch (1995) show the appearance of a ‘turbulent’ diffusion matrix, which is always positive-definite. In order to establish further links we consider the case when the given velocity does not contain high frequency oscillations and can be expressed as $\tilde{\mathbf{u}}_1(\mathbf{x}, t/\omega, t)$ (or just $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_1(\mathbf{x}, t)$) with the 2π -periodic dependence on t . Let us show that in this case the problem can be transformed to the considered in *Sects.2-7*.

Problem A: The dimensionless form of the transport equation (2.10), (2.34) for a purely oscillating velocity

$$\left(\frac{\partial}{\partial s} + \omega^\beta \tilde{\mathbf{u}} \cdot \nabla \right) \hat{a} = 0, \quad \frac{\partial}{\partial s} = \frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \tau} \quad (8.65)$$

where $\hat{a} = \hat{a}(\mathbf{x}, t, \tau)$, $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\mathbf{x}, t, \tau)$, $\beta = \text{const}$, the mutually dependent time-variables are $t = s$, $\tau \equiv \omega s$.

Problem B: A similar equation for different time-scales is

$$\left(\frac{\partial}{\partial s} + \omega^{\beta_1} \tilde{\mathbf{u}}_1 \cdot \nabla \right) \hat{a}_1 = 0, \quad \frac{\partial}{\partial s} = \frac{1}{\omega} \left(\frac{\partial}{\partial t_1} + \omega \frac{\partial}{\partial \tau_1} \right) \quad (8.66)$$

where $\hat{a}_1 = \hat{a}_1(\mathbf{x}, t_1, \tau_1)$, $\tilde{\mathbf{u}}_1 = \tilde{\mathbf{u}}_1(\mathbf{x}, t_1, \tau_1)$, $\beta_1 = \text{const}$, the mutually dependent time-variables are $t_1 = s/\omega$, $\tau_1 \equiv s$.

One can see that the equations (8.65) and (8.66) are mathematically identical to each other if one introduces the link ‘*Problem A* \leftrightarrow *Problem B*’ as

$$t \leftrightarrow t_1, \quad \tau \leftrightarrow \tau_1, \quad \beta \leftrightarrow \beta_1 + 1, \quad \hat{a}_t, \hat{a}_\tau \in \mathcal{O}(1) \leftrightarrow \hat{a}_{t_1}, \hat{a}_{\tau_1} \in \mathcal{O}(1) \quad (8.67)$$

After such replacements any solution of (8.65) simultaneously produces a solution of (8.66). Hence this rescaling procedure delivers a counterpart $\hat{a}(\mathbf{x}, t_1, \tau_1)$ of $\hat{a}(\mathbf{x}, t, \tau)$: in the *Problem B* (8.66) the imposed oscillations have the frequency $\mathcal{O}(1)$ and the slow-time-scale $\mathcal{O}(1/\omega)$. The link (8.67) allows us to add one more solution for each solution in *Sects.2-7*. The main motivation behind the rescaling (8.67) is: the only solutions considered in the homogenisation theory, see Bensoussan, Lions and Papanicolaou (1978), Berdichevsky, Jikov, and Papanikolaou (1999) are of the type (8.66).

The presence of molecular diffusivity. Let μ^* and μ be the dimensional and dimensionless coefficients of molecular diffusion. Eqn. (2.1) describes the advection-diffusion of a scalar admixture in an incompressible fluid after adding the term $\mu^* \nabla^{*2} a$ to the RHS. All results of *Sects.2-6* can be straightforwardly generalized to the flows with $\mu = \mathcal{O}(1)$. Such generalization is achieved by replacing operators $\partial/\partial s^* \rightarrow \partial/\partial s^* - \mu^* \nabla^{*2}$ in (2.1) and $\partial/\partial t \rightarrow \partial/\partial t - \mu \nabla^2$ in (2.4) and in all subsequent formulae. The limit $\mu \rightarrow 0$ is a regular one for our strictly regular asymptotic procedures (4.4), so all the problems with $\mu \neq 0$ have the cases studied in *Sects.2-6* with $\mu \equiv 0$ as their limits.

Magnetohydrodynamics (MHD). One can also develop the same asymptotic procedures and to derive the averaged equations similar to (4.17)-(4.22) for a vectorial passive admixture such as magnetic field $\mathbf{h}^*(\mathbf{x}^*, s^*)$ in the kinematic MHD-dynamo problem for a given oscillating velocity field, see Moffatt (1978). In this case one can start with equations

$$\partial \mathbf{h}^* / \partial s^* + [\mathbf{h}^*, \mathbf{u}^*] = 0, \quad \text{div } \mathbf{h}^* = 0, \quad \text{div } \mathbf{u}^* = 0 \quad (8.68)$$

which replace (2.1). Taking the same velocity field (4.1) and repeating the same steps as

in *Sect.4* lead to the averaged equation (valid with the error $O(\varepsilon^3)$)

$$\begin{aligned} \bar{\mathbf{h}}_t + (\bar{\mathbf{V}} \cdot \nabla) \bar{\mathbf{h}} - (\bar{\mathbf{h}} \cdot \nabla) \bar{\mathbf{V}} &= \frac{\partial}{\partial x_i} \left(\bar{\kappa}_{ik} \frac{\partial \bar{\mathbf{h}}}{\partial x_k} \right), \quad \text{div } \bar{\mathbf{h}} = 0 \\ \bar{\mathbf{h}} &= \bar{\mathbf{h}}_0 + \varepsilon \bar{\mathbf{h}}_1 + \varepsilon^2 \bar{\mathbf{h}}_2, \quad \bar{\mathbf{V}} = \bar{\mathbf{V}}_0 + \varepsilon \bar{\mathbf{V}}_1 + \varepsilon^2 \bar{\mathbf{V}}_2, \quad \bar{\kappa}_{ik} = \varepsilon^2 \bar{\chi}_{ik} \end{aligned} \quad (8.69)$$

where all coefficients are the same as in (4.17)-(4.22). For this problem we have performed the detailed calculations of $\bar{\mathbf{V}}_0$ and $\bar{\mathbf{V}}_1$, while the expressions for $\bar{\mathbf{V}}_2$ and $\bar{\kappa}_{ik}$ represent reliable conjectures. In these calculations one should essentially use (2.24) and (2.32). Notice that eqn. (8.69) describes evolution of $\bar{\mathbf{h}}(\mathbf{x}, t)$ for an arbitrary spatial scale L , while the homogenisation approach to the same problem (see *e.g.* Frisch (1995), Zheligovsky (2009)) operates with long-wave solutions. However, the rescaling of spatial variables (similar to one for time variables in (8.65),(8.66)) is applicable.

9. Discussion and Conclusions

1. The motivation behind this paper is to study in full a variety of asymptotic solutions and procedures for the advection of a scalar or vector field in an oscillating flow. The main result of the paper is the general understanding of the differences between different classes of asymptotic solutions.

2. The author is not aware about any research paper entirely devoted to the general analysis of motions of a scalar (or vector) admixture in high-frequency oscillating flows. At the same time this topic is exploited in many papers in the solutions of particular applied problems (*e.g.* Magar & Pedley (2005)). We hope, that our general study can underpin further applied studies.

3. Several well-known papers are devoted to the motion of a scalar admixture in spatially oscillating flows, *e.g.* McLaughlin, Papanicolaou & Pironneau (1985), Vergassola & Avellaneda (1997), and a review in Frisch (1995). These authors have considered only one asymptotic family of solutions (following to Bensoussan, Lions and Papanicolaou (1978)). Therefore it can be useful to study both high-frequency and short-wave problems at a greater level of generality and then to consider different asymptotic procedures for flows that oscillate both in space and time.

4. In our paper we are dealing with the systematic calculations of solutions. A few restrictions have been accepted at the very beginning, the rest of the paper does not contain any physical input. Hence, our results require physical interpretations, explanations, and applications.

5. The asymptotic validity of our results is provided by the fact that we build only the solutions for high frequency ω . In our approach ω is not linked to any dynamical equations. Instead, we require that dimensional frequency ω^* is high in comparison with the inverted slow-time-scale $1/T$ (2.7) of an arbitrary prescribed velocity field (2.2). If velocity field (2.2) is purely oscillatory, then the choice of T is not unique, which is explained in (2.37),(2.38).

6. All results of this paper have been obtained for the τ -periodic functions from the class \mathbb{H} (2.15), which is closed with respect to all used operations. One can try to consider more complex classes of quasi-periodic, non-periodic, or chaotic solutions. The generalization of our results to quasi-periodic solutions (containing several τ -periods) looks rather straightforward, provided one can deal with resonances. At the same time, chaotic oscillations (containing low ω in their spectrum) can be hardly treated by our method since ω enters the denominators of asymptotic series (the apparent requirement for the applicability of our method is separating the frequency spectrum from zero).

Some generalizations of our results can be achieved after replacing the integration over $\tau_0 < \tau < \tau_0 + 2\pi$ by integration over $-\infty < \tau < +\infty$ in the definition of average (2.17) (as it has been widely accepted for the spatial average in the homogenization theory, see Berdichevsky, Jikov, and Papanikolaou (1999)).

7. In principle, *TTAM* allows to produce the approximate asymptotic solutions with as small an error (RHS-residual) as needed. However, the next logical step is more challenging: one has to prove that a solution with a small RHS-residual is close to the exact one. This problem is equivalent to the presence of a small additional force. Such proofs had been performed by Simonenko (1972), Levenshtam (1996) for vibrational convection. Similar justifications of *TTAM* for other oscillating flows are not available yet.

8. Different asymptotic procedures $\varepsilon_{1/2}$, $\varepsilon_{1/3}$, $\varepsilon_{1/4}$, *etc.* correspond to different asymptotic paths on the plane of two scaling parameters (2.12). Two additional asymptotic paths represent two successive limits: (i) first $\delta \rightarrow 0$ and then $1/\omega \rightarrow 0$ or (ii) first $1/\omega \rightarrow 0$ and then $\delta \rightarrow 0$ (these paths are not considered in this paper but they are worth studying). The relevance of different available paths to particular physical situations represents a major problem for further studies. At this stage we can mention only that different asymptotic paths correspond to different relations between physical parameters. For example, $\delta = \omega^{-1/2}$ (4.1) and $\delta = \omega^{-1/3}$ (5.2) in their dimensional form (2.8) give

$$U = LT^{-1/2}\omega^{*1/2} \quad \text{and} \quad U = LT^{-1/3}\omega^{*2/3} \quad (9.1)$$

correspondingly. For the asymptotic procedures with large ω^* (2.13) it means that the imposed oscillatory velocity (2.2) is higher in the latter (super-critical) case.

9. To be able to compare different flows physically one might introduce a *drift-efficiency* of an oscillatory flow as the ratio

$$E_\alpha \equiv (\text{amplitude of drift velocity})/(\text{amplitude of velocity oscillations})$$

with α defined in (2.9). The dimensionless drifts in both cases (9.1) are $O(1)$ and in dimensional form they are of order L/T . Then one might conclude that the efficiencies of (9.1) are $E_{1/2} = (\omega^*T)^{-1/2}$ and $E_{1/3} = (\omega^*T)^{-2/3}$ correspondingly, which indicates that the critical family solution (4.1) produces stronger drift than the super-critical one (5.2). However, this comparison is not fair since the solution (5.2) is valid only when the leading term in (4.1) vanishes (5.1). Taking the degeneration into account one obtains the efficiencies of (9.1) as

$$E_{1/2}^d = (\omega^*T)^{-1} \quad \text{and} \quad E_{1/3} = (\omega^*T)^{-2/3}$$

where $E_{1/2}^d$ represents the drift-efficiency of the degenerated critical solution (4.1),(5.1). Now, one can conclude that a super-critical solution is more drift-efficient than a critical one.

10. The appearance of small pseudo-diffusion (*PD*) in the high-frequency asymptotic problem (4.2), (4.17)-(4.27) is an accurate and qualitatively new result. One can make two conjectures: (i) a solution can slowly self-concentrate (due to small negative pseudo-diffusivity) or it can undergo unusual anisotropic evolution (due to sign-indefinite pseudo-diffusivity); and (ii) the maximum principle can be violated (since the original equation (2.1) expresses the conservation of \hat{a} in each fluid particle, hence the values of $\sup \hat{a}$ and $\inf \hat{a}$ do not change with time). One can argue that the conjecture (ii) is not valid due to the explanations given in *Sect. 7.10*, while (i) requires additional studies.

11. It is worth introducing the finite molecular diffusivity $\mu = O(1)$ (see *Sect. 8*) that can improve the convergence of all used asymptotic procedures. The introducing of asymptotically small or large diffusivity (*e.g.* $\mu = O(1/\omega)$ or $\mu = O(\omega)$) will generally

lead to different asymptotic theories corresponding to three independent small parameters from the extended list (2.8).

12. In *Sect.7.9* we have shown that the drift, caused by a relatively simple oscillatory velocity, produces chaotic dynamics of particles. This result leads to numerous new questions (see the end of *Sect.7.9*) and deserves a serious elaboration.

13. In *Sect.2.3* we have demonstrated that an infinite and continuous range of slow-time scales $1/\omega^* < T < \infty$ can be used in an important case where slow-time t is absent in the expression for velocity: $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\mathbf{x}, \tau)$. We have formulated this result with the aim to clarify the existence of many scales (which often causes confusions) and to show that an infinite number of similar (to the considered in *Sects.3-7*) solutions are available. At the same time, this result can lead to studying the motions with simultaneous presence of several different scales T , and to developing multi-scale (triple-scale, *etc.*) theories.

14. It is worth completing the calculations of Riemann's invariant and (8.64) and to compare the related solutions for $\hat{\mathbf{u}}$ with that of (4.25). The aim is to find the advantages given by two complementary averaged solutions of the same problem.

15. The version of *TTAM*-theory for a vectorial passive admixture (with the averaged equations (8.69)) is linked with the problem of kinematic *MHD*-dynamo (see Moffatt (1978)) and can bring new results. It is apparent, that for the majority of drift velocities $\bar{\mathbf{V}}(\mathbf{x}, t)$ the stretching of material elements will produce linear growth $|\bar{\mathbf{h}}| \sim t$. At the same time, there are known examples with an exponential stretching of material lines with time, say, in the flows near stagnation points or in chaotic flows. These examples will provide the exponential growth of $|\bar{\mathbf{h}}|$.

16. Our theory brings up a new possibility: one can find an oscillatory velocity $\tilde{\mathbf{u}}$ which produces any required drift field (for example, a drift in the form of ABC-flow). The results of *Sects.4-7* (including (7.7),(7.38)) indicate that the solution of this problem is not unique.

17. *TTAM*-method and results have potential applications in the studies of a broad variety of flows. The discovery of drift motions with the velocities $\bar{\mathbf{V}}_1 = O(1)$ and $\bar{\mathbf{V}}_2 = O(1)$ can lead to new applications. The other advantages of our method, that can be exploited in applications, are: (i) the mathematical generality of the imposed oscillatory velocity fields (2.2),(6.1); (ii) the wide class of scalings (2.14); and (iii) the most straightforward Eulerian average. In particular, our method can be useful for applications to biologically motivated flows possessing complex geometry (*e.g.* Hydon & Pedley (1993), Magar & Pedley (2005), Leptos, Guasto, Gollub, Pesci, and Goldstein (2009)), for microhydrodynamics (*e.g.* Leal (1980), Kim & Karriba (1991), Hinch & Nitsche (1993), Moffatt (1996)); for flow mixing (*e.g.* Ottino (1989), Carlsson, Sen, and Lofdahl (2005), Carlsson, Sen, and Lofdahl (2004), Leal (2007)), for the flow of blood and flows in lungs, see Pedley (1980), and for the pipes with oscillating walls, see Carpenter & Pedley (2001). The spreading of ash from recently erupted Eyjafjallajökull volcano has made the research on the general area of the transport of an admixture in a fluid (which can oscillate due to various reasons) even more important.

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Appendix A. Calculations for $\alpha = 1/2$ of Sect.4

The zero-order equation (4.5) is:

$$\hat{a}_{0\tau} = 0 \quad (\text{A } 1)$$

The substitution of $\hat{a}_0 = \bar{a}_0(\mathbf{x}, t) + \tilde{a}_0(\mathbf{x}, t, \tau)$ into (A 1) gives $\tilde{a}_{0\tau} = 0$. Its \mathbb{T} -integration (2.21) produces a unique (inside the \mathbb{T} -class) solution $\tilde{a}_0 \equiv 0$. At the same time (A 1) does not impose any restrictions on $\bar{a}_0(\mathbf{x}, t)$, which must be determined from the next approximations. Thus the results derivable from (A 1) are:

$$\tilde{a}_0(\mathbf{x}, t, \tau) \equiv 0, \quad \forall \bar{a}_0 = \bar{a}_0(\mathbf{x}, t); \quad \hat{a}^{[0]} \equiv \bar{a}_0 + \tilde{a}_0 = \bar{a}_0(\mathbf{x}, t) \quad (\text{A } 2)$$

The substitution of the truncated solution $\hat{a}^{[0]}$ into the governing equation (4.3) produces the RHS-residual of order ε :

$$\mathfrak{D}_2 \hat{a}^{[0]} = \text{Res}[0] \equiv \varepsilon(\tilde{\mathbf{u}} \cdot \nabla) \bar{a}_0 + \varepsilon^2 \bar{a}_{0t} = O(\varepsilon) \quad (\text{A } 3)$$

The first-order equation (4.6) is

$$\hat{a}_{1\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_0 \quad (\text{A } 4)$$

The use of $\tilde{a}_0 \equiv 0$ (A 2) and $\bar{a}_{1\tau} \equiv 0$ reduces (A 4) to the equation $\tilde{a}_{1\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \bar{a}_0$. Its \mathbb{T} -integration (2.21) gives the unique solution for \tilde{a}_1

$$\tilde{a}_1 = -(\tilde{\xi} \cdot \nabla) \bar{a}_0 \quad (\text{A } 5)$$

where $\tilde{\xi} \equiv \tilde{\mathbf{u}}^\tau$ (4.16). Hence, \hat{a}_1 and $\hat{a}^{[1]}$ are

$$\hat{a}_1 = \bar{a}_1 - (\tilde{\xi} \cdot \nabla) \bar{a}_0, \quad \hat{a}^{[1]} = \bar{a}_0 + \varepsilon \hat{a}_1; \quad \forall \bar{a}_0(\mathbf{x}, t) \text{ and } \bar{a}_1(\mathbf{x}, t) \quad (\text{A } 6)$$

The substitution of $\hat{a}^{[1]}$ into (4.3) produces the RHS-residual of order ε^2 :

$$\mathfrak{D}_2 \hat{a}^{[1]} = \text{Res}[1] \equiv \varepsilon^2 \left((\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_1 + \partial_t \hat{a}^{[1]} \right) = O(\varepsilon^2) \quad (\text{A } 7)$$

Formulae (A 2) and (A 5) give (4.11) and (4.12).

The second-order equation ((4.7) for $n = 2$) is

$$\hat{a}_{2\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_1 - \hat{a}_{0t} \quad (\text{A } 8)$$

The use of (A 2) and $\bar{a}_{2\tau} \equiv 0$ transforms (A 8) to

$$\tilde{a}_{2\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \bar{a}_1 - (\tilde{\mathbf{u}} \cdot \nabla) \tilde{a}_1 - \bar{a}_{0t} \quad (\text{A } 9)$$

Its \mathbb{B} -part is

$$\bar{a}_{0t} = -\langle (\tilde{\mathbf{u}} \cdot \nabla) \tilde{a}_1 \rangle \quad (\text{A } 10)$$

where we have used $\langle \tilde{a}_{2\tau} \rangle = 0$, $\langle (\tilde{\mathbf{u}} \cdot \nabla) \bar{a}_1 \rangle = 0$, and $\langle \bar{a}_{0t} \rangle = \bar{a}_{0t}$. The substitution of (A 5) into (A 10) produces the equation

$$\bar{a}_{0t} = \langle (\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle \bar{a}_0 \quad (\text{A } 11)$$

which represents a version of inspection equation (3.13) systematically derived. One may expect that the RHS of (A 11) contains both first and second spatial derivatives of \bar{a}_0 , however *all second derivatives vanish*. In order to prove it we introduce a commutator (2.23)

$$\widehat{\mathbf{K}} \equiv [\tilde{\xi}, \tilde{\mathbf{u}}] = (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\xi} - (\tilde{\xi} \cdot \nabla) \tilde{\mathbf{u}}, \quad (\text{A } 12)$$

$$(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\xi} \cdot \nabla) - (\tilde{\xi} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla) = \widehat{\mathbf{K}} \cdot \nabla. \quad (\text{A } 13)$$

The \mathbb{B} -part of (A 13) is

$$\langle(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle = \langle(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)\rangle + \overline{\mathbf{K}} \cdot \nabla \quad (\text{A } 14)$$

At the same time the integration by parts (2.29) gives

$$\langle(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle = -\langle(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)\rangle, \quad \tilde{\mathbf{u}} \equiv \tilde{\boldsymbol{\xi}}_\tau \quad (\text{A } 15)$$

Combining (A 14) and (A 15) we obtain

$$\langle(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle = \frac{1}{2} \overline{\mathbf{K}} \cdot \nabla \quad (\text{A } 16)$$

which reduces (A 11) to the advection equation (4.17) with

$$\overline{\mathbf{V}}_0 \equiv -\langle(\tilde{\mathbf{u}} \cdot \nabla) \tilde{\boldsymbol{\xi}}\rangle = -\frac{1}{2} \langle[\tilde{\boldsymbol{\xi}}, \tilde{\mathbf{u}}]\rangle = -\frac{1}{2} \overline{\mathbf{K}} \quad (\text{A } 17)$$

which coincides with (4.20). The \mathbb{T} -part of (A 9) appears after subtracting (A 10) from (A 9) and the use of notation (2.28):

$$\tilde{a}_{2\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \bar{a}_1 - \{(\tilde{\mathbf{u}} \cdot \nabla) \tilde{a}_1\}. \quad (\text{A } 18)$$

Its \mathbb{T} -integration (2.21) with the use of (A 5) gives (4.13):

$$\tilde{a}_2 = -(\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_1 + \{(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\}^\tau \bar{a}_0, \quad \forall \bar{a}_1 \quad (\text{A } 19)$$

Hence, \hat{a}_2 and $\hat{a}^{[2]}$ can be written as

$$\hat{a}_2 = \bar{a}_2 + \tilde{a}_2, \quad \hat{a}^{[2]} = \bar{a}_0 + \varepsilon \left(\bar{a}_1 - (\tilde{\boldsymbol{\xi}} \cdot \nabla) \bar{a}_0 \right) + \varepsilon^2 \hat{a}_2, \quad \forall \bar{a}_1, \bar{a}_2 \quad (\text{A } 20)$$

where \bar{a}_0 and \tilde{a}_2 are given by (4.17), (A 19). The substitution of $\hat{a}^{[2]}$ into (4.3) produces the RHS-residual of order ε^3

$$\mathfrak{D}_2 \hat{a}^{[2]} = \text{Res}[2] \equiv \varepsilon^3 ((\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_2 + \partial_t (\hat{a}_1 + \varepsilon \hat{a}_2)) = O(\varepsilon^3). \quad (\text{A } 21)$$

The third-order equation ((4.7) for $n = 3$) is:

$$\tilde{a}_{3\tau} = -(\tilde{\mathbf{u}} \cdot \nabla) \hat{a}_2 - \hat{a}_{1t} \quad (\text{A } 22)$$

Its \mathbb{B} -part is

$$\bar{a}_{1t} = -\langle(\tilde{\mathbf{u}} \cdot \nabla) \tilde{a}_2\rangle. \quad (\text{A } 23)$$

The substitution of (A 19) into (A 23), the use of $\tilde{\mathbf{u}} \equiv \tilde{\boldsymbol{\xi}}_\tau$, and the integration by parts (2.29) yield

$$\bar{a}_{1t} = \langle(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle \bar{a}_1 + \langle(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle \bar{a}_0 \quad (\text{A } 24)$$

where $\langle(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle$ has been already simplified in (A 16). The second term in the RHS of (A 24) formally contains the third, second, and first spatial derivatives of \bar{a}_0 ; however *all the third and second derivatives vanish*. To prove it, first, we use (2.30):

$$\langle(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle = -\langle(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle - \langle(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)\rangle \quad (\text{A } 25)$$

Then we use (A 12), (A 13) to transform the sequence of operators $(\tilde{\boldsymbol{\xi}} \cdot \nabla)$ and $(\tilde{\mathbf{u}} \cdot \nabla)$ in each term in the RHS of (A 25) into their sequence in the LHS. The result is

$$\langle(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)\rangle = \frac{1}{3} \overline{\mathbf{K}}' \cdot \nabla, \quad \widehat{\mathbf{K}}' \equiv [\widehat{\mathbf{K}}, \tilde{\boldsymbol{\xi}}] \quad (\text{A } 26)$$

As the result (A 24) takes form (4.18) with $\overline{\mathbf{V}}_0$ (A 17) and

$$\overline{\mathbf{V}}_1 \equiv -\langle (\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\boldsymbol{\xi}} \rangle = -\frac{1}{3}\langle [[\tilde{\boldsymbol{\xi}}, \tilde{\mathbf{u}}], \tilde{\boldsymbol{\xi}}] \rangle = -\frac{1}{3}\overline{\mathbf{K}}^T \quad (\text{A } 27)$$

which gives (4.20). The \mathbb{T} -part of (A 22) after its \mathbb{T} -integration gives (4.14)

$$\tilde{a}_3 = -(\tilde{\boldsymbol{\xi}} \cdot \nabla)\tilde{a}_2 - \{(\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_2\}^T - \tilde{a}_{1t}^T, \quad \tilde{a}_1^T = -(\tilde{\boldsymbol{\xi}}^T \cdot \nabla)\tilde{a}_0 \quad (\text{A } 28)$$

where \tilde{a}_2 is given by (A 19).

Hence, \hat{a}_3 and $\hat{a}^{[3]}$ can be written as

$$\hat{a}_3 = \bar{a}_3 + \tilde{a}_3, \quad \hat{a}^{[3]} = \bar{a}_0 + \varepsilon(\bar{a}_1 - (\tilde{\boldsymbol{\xi}} \cdot \nabla)\bar{a}_0) + \varepsilon^2\hat{a}_2 + \varepsilon^3\hat{a}_3, \quad \forall \bar{a}_2, \bar{a}_3 \quad (\text{A } 29)$$

where \bar{a}_0 , \bar{a}_1 , \tilde{a}_2 , and \tilde{a}_3 are given by (4.17), (4.18), (A 19), and (A 28). The substitution of $\hat{a}^{[3]}$ into (4.3) produces the RHS-residual of order ε^4 :

$$\mathfrak{D}_2\hat{a}^{[3]} = \text{Res}[3] = O(\varepsilon^4) \quad (\text{A } 30)$$

Explicit formulae for residuals (similar to (A 3), (A 7), (A 21)) for (A 30), (4.29) and for all other considered cases can be calculated straightforwardly; however for brevity we do not present them.

The fourth-order equation ((4.7) for $n = 4$) is:

$$\tilde{a}_{4\tau} = -(\tilde{\mathbf{u}} \cdot \nabla)\hat{a}_3 - \hat{a}_{2t} \quad (\text{A } 31)$$

Its \mathbb{B} -part is

$$\bar{a}_{2t} = -\langle (\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_3 \rangle \quad (\text{A } 32)$$

The substitution of \tilde{a}_3 (A 28) into (A 32), \tilde{a}_2 (A 19) into \tilde{a}_3 (A 28), the integration by parts (2.29), and the use of (4.16), (A 17), (A 27) yield

$$\begin{aligned} \langle (\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_3 \rangle &= (\overline{\mathbf{V}}_0 \cdot \nabla)\bar{a}_2 + (\overline{\mathbf{V}}_1 \cdot \nabla)\bar{a}_1 + \langle (\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \rangle (\overline{\mathbf{V}}_0 \cdot \nabla)\bar{a}_0 - \\ &- \langle (\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}}_t \cdot \nabla) \rangle \bar{a}_0 + \overline{\mathfrak{X}}\bar{a}_0, \quad \text{where } \overline{\mathfrak{X}} \equiv \langle (\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla) \{ (\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \}^T \rangle \end{aligned} \quad (\text{A } 33)$$

The shorthand operator $\overline{\mathfrak{X}}$ (as well as the operators $\overline{\mathfrak{Y}}$, $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$, $\overline{\mathfrak{C}}$, and $\overline{\mathfrak{F}}$ below) acts on \bar{a}_0 . The RHS of (A 33) formally contains the fourth, third, second, and first spatial derivatives of \bar{a}_0 ; however *all the fourth and third derivatives vanish*. In order to prove it we first rewrite $\overline{\mathfrak{X}}$ as

$$\overline{\mathfrak{X}} = \langle (\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)\overline{\mathfrak{Y}}^T \rangle \quad \text{where} \quad \overline{\mathfrak{Y}} \equiv (\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \quad (\text{A } 34)$$

The use of (2.30) and (A 12), (A 13) transforms (A 34) to

$$2\overline{\mathfrak{X}} = -\overline{\mathfrak{A}} + \overline{\mathfrak{B}} + \frac{1}{2}\langle (\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \rangle (\overline{\mathbf{K}} \cdot \nabla) \quad (\text{A } 35)$$

$$\overline{\mathfrak{A}} \equiv \langle (\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \rangle, \quad \overline{\mathfrak{B}} \equiv \langle (\widetilde{\mathbf{K}}^T \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \rangle$$

Let us now simplify $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{B}}$. For $\overline{\mathfrak{B}}$ we use (2.30)

$$\overline{\mathfrak{B}} \equiv \langle (\widetilde{\mathbf{K}}^T \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \rangle = -\langle (\widetilde{\mathbf{K}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \rangle - \langle (\widetilde{\mathbf{K}}^T \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla) \rangle$$

To change $(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)$ into $(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)$ in the last term we use (A 12), (A 13) that yields:

$$\overline{\mathfrak{B}} = -\frac{1}{2}\langle (\widetilde{\mathbf{K}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla)(\tilde{\boldsymbol{\xi}} \cdot \nabla) \rangle - \frac{1}{4}\widehat{\boldsymbol{\kappa}} \cdot \nabla, \quad \widehat{\boldsymbol{\kappa}} \equiv [\widetilde{\mathbf{K}}^T, \widetilde{\mathbf{K}}] \quad (\text{A } 36)$$

The operator $\overline{\mathfrak{A}}$ is simplified by the version of (2.30) with four multipliers

$$\begin{aligned} \overline{\mathfrak{A}} \equiv & \langle (\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle = -\langle (\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle - \\ & -\langle (\tilde{\xi} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle - \langle (\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla)(\tilde{\mathbf{u}} \cdot \nabla) \rangle \end{aligned} \quad (\text{A } 37)$$

The multiple use of commutator (A 12), (A 13) allows us to transform the sequence of operators $(\tilde{\xi} \cdot \nabla)$ and $(\tilde{\mathbf{u}} \cdot \nabla)$ in each term in the RHS of (A 37) to the sequence in its LHS. The result is

$$\overline{\mathfrak{A}} = -\frac{1}{2}\langle (\widehat{\mathbf{K}} \cdot \nabla)(\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle + \frac{1}{4}\overline{\mathbf{K}''} \cdot \nabla, \quad \widehat{\mathbf{K}''} \equiv [\widehat{\mathbf{K}'}, \tilde{\xi}] \quad (\text{A } 38)$$

Now, (A 35), (A 36), and (A 38) yield

$$\overline{\mathfrak{X}} = \frac{1}{4}(\overline{\mathbf{K}} \cdot \nabla)\langle (\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle + \frac{1}{4}\langle (\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle(\overline{\mathbf{K}} \cdot \nabla) - \frac{1}{8}(\overline{\mathbf{K}} + \overline{\mathbf{K}''}) \cdot \nabla$$

The substitution of this expression into (A 33), (A 32) gives

$$\begin{aligned} \overline{a}_{2t} + (\overline{\mathbf{V}}_0 \cdot \nabla)\overline{a}_2 + (\overline{\mathbf{V}}_1 \cdot \nabla)\overline{a}_1 - \frac{1}{8}(\overline{\mathbf{K}} + \overline{\mathbf{K}''}) \cdot \nabla\overline{a}_0 + \frac{1}{4}\overline{\mathfrak{C}}\overline{a}_0 - \overline{\mathfrak{F}}\overline{a}_0 &= 0, \quad (\text{A } 39) \\ \overline{\mathfrak{C}} \equiv (\overline{\mathbf{K}} \cdot \nabla)\langle (\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle - \langle (\tilde{\xi} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle(\overline{\mathbf{K}} \cdot \nabla), \quad \overline{\mathfrak{F}} \equiv \langle (\tilde{\xi} \cdot \nabla)(\tilde{\xi}_t \cdot \nabla) \rangle \end{aligned}$$

Additional transformations of the last two operators in (A 39) yield

$$\begin{aligned} \frac{1}{4}\overline{\mathfrak{C}} - \overline{\mathfrak{F}} &= \frac{1}{2}\langle [\tilde{\xi}, \tilde{\xi}_t] \rangle \cdot \nabla - \frac{1}{2}\langle (\tilde{\mathbf{u}}' \cdot \nabla)\tilde{\xi} + (\tilde{\xi} \cdot \nabla)\tilde{\mathbf{u}}' \rangle \cdot \nabla - \frac{1}{2}\langle \tilde{u}'_i \tilde{\xi}_k + \tilde{u}'_k \tilde{\xi}_i \rangle \frac{\partial^2}{\partial x_i \partial x_k} = \\ &= \frac{1}{2}\langle [\tilde{\xi}, \tilde{\xi}_t] \rangle \cdot \nabla - \frac{\partial}{\partial x_k} \left(\overline{\chi}_{ik} \frac{\partial}{\partial x_i} \right) + \frac{1}{2}\langle \tilde{\xi} \text{div } \tilde{\mathbf{u}}' + \tilde{\mathbf{u}}' \text{div } \tilde{\xi} \rangle \end{aligned} \quad (\text{A } 40)$$

$$\tilde{\mathbf{u}}' \equiv \tilde{\xi}_t - [\overline{\mathbf{V}}_0, \tilde{\xi}], \quad \overline{\chi}_{ik} \equiv \frac{1}{2}\langle \tilde{u}'_i \tilde{\xi}_k + \tilde{u}'_k \tilde{\xi}_i \rangle \quad (\text{A } 41)$$

The substitution of (A 40) into (A 39) leads to the equation for \overline{a}_2 (4.19) where the formula (4.23) for $\overline{\chi}_{ik}$ is obtained from (A 41) by the use of definition $\tilde{\mathbf{u}}'$.

The \mathbb{T} -part of (A 31) after its \mathbb{T} -integration gives (4.15)

$$\tilde{a}_4 = -(\tilde{\xi} \cdot \nabla)\tilde{a}_3 - \{(\tilde{\mathbf{u}} \cdot \nabla)\tilde{a}_3\}^\tau - \tilde{a}_{2t}^\tau \quad (\text{A } 42)$$

where \tilde{a}_3 is given by (A 28).

Appendix B. Calculations for Super-critical Families

B.1. Calculations for $\alpha = 1/3$ of Sect.5

The equations of the *zeroth and first approximations* (5.5), (5.6) are the same as (4.5), (4.6) hence they have the same solutions

$$\widehat{a}_0 = \overline{a}_0(\mathbf{x}, t), \quad \widehat{a}_1 = \overline{a}_1 - (\tilde{\xi} \cdot \nabla)\overline{a}_0; \quad \forall \overline{a}_0, \overline{a}_1 \in \mathbb{B} \cap \mathbb{O}(1) \quad (\text{B } 1)$$

An essential difference from the $\omega^{1/2}$ -procedure emerges for the *second approximation* (5.7) where instead of an advection equation for \overline{a}_0 (4.17) we obtain a compatibility condition

$$(\overline{\mathbf{V}}_0 \cdot \nabla)\overline{a}_0 = 0 \quad (\text{B } 2)$$

which represents an identity due to (5.1). The \mathbb{T} -part of (5.7) produces the same \tilde{a}_2 as in (A 19). Hence

$$\widehat{a}_2 = \overline{a}_2 + \tilde{a}_2, \quad \tilde{a}_2 = -(\tilde{\xi} \cdot \nabla)\overline{a}_1 + \{(\tilde{\mathbf{u}} \cdot \nabla)(\tilde{\xi} \cdot \nabla)\}^\tau \overline{a}_0, \quad \forall \overline{a}_2 \in \mathbb{B} \cap \mathbb{O}(1) \quad (\text{B } 3)$$

The equation for the *third approximation* ($n = 3$ in (5.8)) jointly with (5.1) produces the \mathbb{B} -part of the equation

$$\bar{a}_{0t} = \langle (\tilde{\xi} \cdot \nabla)(\tilde{u} \cdot \nabla)(\tilde{\xi} \cdot \nabla) \rangle \bar{a}_0 \quad (\text{B } 4)$$

with the same triple-correlation as in (A 26), (A 27); hence

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{V}}_1 \cdot \nabla \right) \bar{a}_0 = 0, \quad \text{where} \quad \bar{\mathbf{V}}_1 = -\frac{1}{3} \langle [[\tilde{\xi}, \tilde{u}], \tilde{\xi}] \rangle \quad (\text{B } 5)$$

The \mathbb{T} -part of (5.8) after its \mathbb{T} -integration gives a simpler expression

$$\tilde{a}_3 = -(\tilde{\xi} \cdot \nabla) \bar{a}_2 - \{(\tilde{u} \cdot \nabla) \bar{a}_2\}^\tau \quad (\text{B } 6)$$

with \tilde{a}_2 (B 3). Hence, \hat{a}_3 and $\hat{a}^{[3]}$ can be written as

$$\hat{a}_3 = \bar{a}_3 + \tilde{a}_3, \quad \hat{a}^{[3]} = \bar{a}_0 + \varepsilon(\bar{a}_1 - (\tilde{\xi} \cdot \nabla) \bar{a}_0) + \varepsilon^2 \hat{a}_2 + \varepsilon^3 \hat{a}_3 \quad (\text{B } 7)$$

where \bar{a}_0 , \tilde{a}_2 , and \tilde{a}_3 are given by (B 5), (B 3), and (B 6), while \bar{a}_1 , \bar{a}_2 and \bar{a}_3 are to be found from further approximations. Hence the $\omega^{1/3}$ -procedure shows that in the presence of degeneration (5.1) a drift velocity remains $O(1)$ and is given by the expression $\bar{\mathbf{V}}_1$ (B 5).

The equation for the *fourth approximation* ((5.8) with $n = 4$) is:

$$\tilde{a}_{4\tau} = -(\tilde{u} \cdot \nabla) \hat{a}_3 - \hat{a}_{1t} \quad (\text{B } 8)$$

After the use of the same steps as in the $\omega^{1/2}$ -procedure we have \mathbb{B} -part

$$\bar{a}_{1t} + (\bar{\mathbf{V}}_1 \cdot \nabla) \bar{a}_1 + (\bar{\mathbf{V}}_2 \cdot \nabla) \bar{a}_0 = 0, \quad (\text{B } 9)$$

with the same notations as in (B 5), (5.17), (5.18). The \mathbb{T} -part of (B 8) after its \mathbb{T} -integration gives

$$\tilde{a}_4 = -(\tilde{\xi} \cdot \nabla) \bar{a}_3 - \{(\tilde{u} \cdot \nabla) \tilde{a}_3\}^\tau - \tilde{a}_{1t}^\tau \quad (\text{B } 10)$$

with \tilde{a}_3 (B 6).

B.2. Calculations for $\alpha = 1/4$ of Sect.5

The equation of the *zeroth, first, and second approximations* (5.24), (5.25), and (5.26) are the same as (5.5), (5.6), and (5.7) so they have the same solutions (B 1)-(B 3)

$$\hat{a}_0 \equiv \bar{a}_0 + \tilde{a}_0 = \bar{a}_0(\mathbf{x}, t), \quad \hat{a}_1 = \bar{a}_1 - (\tilde{\xi} \cdot \nabla) \bar{a}_0, \quad \hat{a}_2 = \bar{a}_2 + \tilde{a}_2 \quad (\text{B } 11)$$

An essential difference from the $\omega^{1/3}$ -procedure emerges for the *third approximation* (5.26) where instead of an advection equation for \bar{a}_0 (B 5) we obtain a compatibility condition

$$(\bar{\mathbf{V}}_1 \cdot \nabla) \bar{a}_0 = 0 \quad (\text{B } 12)$$

which represents an identity due to (5.20). The \mathbb{T} -part of (5.27) produces the same \tilde{a}_3 as in (B 6). The equation for the *fourth approximation* ((5.28) with $n = 4$) is:

$$\tilde{a}_{4\tau} = -(\tilde{u} \cdot \nabla) \hat{a}_3 - \hat{a}_{0t} \quad (\text{B } 13)$$

After performing the same steps as in the $\omega^{1/3}$ -procedure we obtain \mathbb{B} -part

$$\bar{a}_{0t} + (\bar{\mathbf{V}}_2 \cdot \nabla) \bar{a}_0 = 0, \quad (\text{B } 14)$$

with the same notations as in (5.17), (5.18). The \mathbb{T} -part of (B 13) after its \mathbb{T} -integration gives

$$\tilde{a}_4 = -(\tilde{\xi} \cdot \nabla) \bar{a}_3 - \{(\tilde{u} \cdot \nabla) \tilde{a}_3\}^\tau \quad (\text{B } 15)$$

where \tilde{a}_3 is given in (B 6).

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